INSTABILITY OF SOME BGK WAVES FOR THE VLASOV-POISSON SYSTEM

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ABSTRACT. We consider a one-dimensional, collisionless plasma given by the Vlasov-Poisson system and investigate the stability of periodic steady state solutions known as Bernstein-Greene-Kruskal (BGK) waves. Using information about the induced dispersion operator, we determine sufficient conditions under which BGK waves are linearly unstable under perturbations that share the same period as the equilibria. We also prove that such solutions cannot support a monotonically decreasing distribution function.

1. INTRODUCTION

A plasma is a partially or completely ionized gas. Such a form of matter occurs if the velocity of individual particles in a material achieves an enormous magnitude, perhaps a sizable fraction of the speed of light. Hence all matter, if heated to a significantly great temperature, will transform into a plasma phase. When a plasma is of low density or the time scales of interest are sufficiently small, it is deemed to be “collisionless”, as collisions between particles become infrequent. Many examples of collisionless plasmas occur in nature, including the solar wind, galactic nebulae, and comet tails.

The fundamental one-dimensional model of collisionless plasma dynamics is given by a system of partial differential equations known as the Vlasov-Poisson system:

\[
\begin{align*}
\frac{\partial}{\partial t} f + v \frac{\partial}{\partial x} f - E \frac{\partial}{\partial v} f &= 0 \\
\frac{\partial}{\partial x} E &= 1 - \int f \, dv.
\end{align*}
\]

(VP)

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Here \( f = f(t, x, v) \) represents the distribution of electrons in the plasma, while \( E = E(t, x) \) is the self-consistent electric field generated by ions and electrons. The independent variables, \( t > 0 \) and \( x, v \in \mathbb{R} \) represent time, position, and velocity, respectively. Instead of studying a large collection of ionic species interacting with the electrons, the density of ions is given by a neutralizing background, normalized to 1 in the equations above. For a general reference regarding the kinetic equations of plasma dynamics, including (VP) and its electromagnetic counterpart, the Vlasov-Maxwell system, we mention \([3]\) and \([12]\).

In the current study we are interested in the stability properties of steady state, or time-independent, solutions of the system. More specifically, we wish to study well-known steady states called BGK waves. In 1957 Bernstein, Greene, and Kruskal \([2]\) showed the existence of an infinite family of exact stationary solutions to (VP) that have come to be known as BGK waves. Since then, the stability or instability of these solutions has been of great interest to the plasma physics and applied mathematics communities \([4, 5, 6, 11, 10]\). More recently, there has been renewed interest in the theory of BGK waves due in part to recent experimental identifications of electrostatic solitary waves in space plasmas. A BGK wave is a steady solution of the form

\[
\begin{cases}
\hat{f}(x, v) = \mu \left( \frac{1}{2} v^2 + \phi(x) \right) \\
\hat{E}(x) = \phi'(x)
\end{cases}
\]

where \( \phi \) is a periodic solution of

\[
\phi''(x) = 1 - \int \mu \left( \frac{1}{2} v^2 + \phi(x) \right) dv.
\]

In \([9]\) the stability properties of such solutions to (VP) were studied. There it was determined that any BGK wave is linearly unstable with respect to multi-periodic perturbations, that is, perturbations whose periods are integer multiples (2 or greater) of the period of the steady state \( \phi \). Unfortunately, such a general result in the singly-periodic case, in which the perturbation and BGK wave possess the same period, remains unknown and does not seem likely. As the author states in \([9]\), one cannot hope that all periodic BGK waves are unstable with respect to such perturbations. Instead, the stability properties may depend delicately on the specific BGK wave of interest. To date little information regarding this question is known, though some results concerning weakly-inhomogeneous solutions have been obtained \([6, 11]\). The main focus of our paper, then, is to determine sufficient conditions under which these BGK waves \((\mu, \phi)\) are linearly unstable due to singly-periodic perturbations.

As Lin constructed a strong framework for studying the problem, we will follow many of the assumptions and techniques developed in \([9]\). To be
precise, we make the following assumptions on the BGK waves (1.1) - (1.2) throughout:

(i) $\mu \in C^1(\mathbb{R})$ is nonnegative.

(ii) $\mu$ satisfies the condition of neutrality, i.e.
$$\int \mu \left( \frac{1}{2} v^2 \right) dv = 1.$$

(iii) $\mu'$ decays at infinity, i.e. there is $\gamma > 1$ and $C > 0$ such that
$$|\mu'(y)| \leq \frac{C}{1 + |y|^\gamma}.$$

(iv) We arrange the period of $\phi$ in a specific manner. Let $P_\phi$ be the minimal period of the solution $\phi$ from (1.2) so that
$$\phi(x) = \phi(x + P_\phi)$$
for every $x \in \mathbb{R}$ and define the quantities
$$\phi_- = \min_{x \in [0,P_\phi]} \phi(x),$$
$$\phi_+ = \max_{x \in [0,P_\phi]} \phi(x).$$

Then, without loss of generality we may rearrange the starting point to impose conditions on the values of $\phi$ so that it satisfies:
$$\phi(0) = \phi(P_\phi) = \phi_+,$$
$$\phi \left( \frac{P_\phi}{2} \right) = \phi_-,$$
$$\phi(x) = \phi(P_\phi - x), \quad \forall x \in [0,P_\phi].$$

Additionally, without loss of generality we can take $\phi$ to be strictly decreasing on the interval $\left[ 0, \frac{P_\phi}{2} \right]$.

With this structure in place, we will prove that (1.2) cannot possess solutions for which $\mu$ is strictly decreasing. We will describe later why this may have interesting consequences in analyzing the stability or instability of such solutions. More importantly, we will show in subsequent sections that, while one cannot hope that all such solutions to (1.1)-(1.2) are linearly unstable with respect to $P_\phi$-periodic perturbations, conditions do exist which guarantee their instability at the linear level. To be more precise further definitions are needed. For $\mu$ given above define the function
$$q(x) = \int \mu' \left( \frac{1}{2} v^2 + \phi(x) \right) dv$$
and let $\lambda_0$ be the smallest eigenvalue, with corresponding first eigenvector $\psi_0 \in H^2(0,P_\phi)$, of the problem
$$\psi''(x) + (q(x) + \lambda)\psi(x) = 0$$
with $\psi(0) = \psi(P_\phi) = 0$. We will prove that if $q$ satisfies

$$(v) \int_0^1 P_\phi (q(x) + \lambda_0)(q(x) + \frac{4}{3}\lambda_0)|\psi_0(x)|^4 \, dx < 0$$

the BGK wave $(\mu, \phi)$ must be linearly unstable with respect to perturbations of period $P_\phi$. Our main theorems will be stated precisely in the next section. The paper proceeds as follows. In Section 2 we will state numerous lemmas and use them to prove the main results. Sections 3 and 4 will then contain the proofs of the aforementioned lemmata. Throughout the paper the value $C > 0$ will denote a generic constant that may change from line to line. When necessary, we will specifically identify the quantities upon which $C$ may depend. Throughout we will also make frequent use of the following spaces of periodic functions that are defined on the real line:

$$\mathcal{L} := \left\{ u \in L^2(0, P_\phi) : \int_0^{P_\phi} u(x) \, dx = 0 \text{ and } \forall x \in \mathbb{R}, \ u(x) = u(x + P_\phi) \right\},$$

$$H^k_{\text{per}}(0, P_\phi) := \left\{ u \in \mathcal{L} : u^{(j)} \in \mathcal{L} \text{ for every } j \in \mathbb{N} \text{ with } j \leq k \right\},$$

$$\mathcal{H} := H^2_{\text{per}}(0, P_\phi).$$

2. Dispersion Operator and Main Theorems

To begin this section, we first derive the system of PDEs that perturbations of (VP) must satisfy so that we may study their behavior. To determine the stability properties of these solutions we consider the initial value problem with time dependent perturbations of the form:

$$f(t, x, v) = \mu \left( \frac{1}{2}v^2 + \phi(x) \right) + F(t, x, v)$$

$$E(t, x) = \phi'(x) + \beta(t, x).$$

Using these quantities in (VP) yields equations for the perturbations, namely

$$\partial_t F + v \phi' \mu' + v \partial_x F - (\phi' + \beta)v\mu' - (\phi' + \beta)\partial_v F = 0$$

and

$$\partial_x \beta - \phi'' = 1 - \int \mu(v)dv - \int F(t, x, v)dv.$$ 

Since $(\mu, \phi)$ is a time-independent solution to (VP), these equations simplify and we arrive at

$$\begin{cases} 
\partial_t F + v \partial_x F - \phi' \partial_v F = v\beta \mu' + \beta \partial_v F \\
\partial_x \beta = - \int Fdv.
\end{cases} \tag{2.1}$$

Notice that the system of partial differential equations (2.1) is nonlinear due to the appearance of the $\beta \partial_v F$ term. We focus on studying the linearized system, so we remove this term, which yields
for unknown perturbations of both the distribution function $F(t,x,v)$ and the field $\beta(t,x)$. This linearized system of PDE then constitutes the equations we wish to study. The main question is whether solutions $(F, \beta)$ tend to zero over long times, thereby leading the solution $(f, E)$ of (VP) to tend to the BGK waves (stability), or whether solutions to (2.2) may actually grow or remain large over time, thereby causing $f$ and $E$ to stay far from the equilibria (instability). Our main result concerning the linear instability of periodic BGK waves (1.1) - (1.2), is stated precisely in the following

**Theorem 2.1.** Let $\mathcal{f}$ and $\mathcal{E}$ be the periodic BGK solutions of (VP) described above with period $P_\phi$. Assume conditions (i) - (v) hold, then there exists $\lambda > 0$ and a solution of the form

$$F(t,x,v) = e^{\lambda t} F(x,v)$$

$$\beta(t,x) = e^{\lambda t} \Psi(x)$$

to the linearized perturbation equations

$$\left\{\begin{array}{l}
\partial_t F + v \partial_x F - \phi' \partial_v F = \beta \partial_v \mathcal{f} \\
\partial_x \beta = - \int F dv.
\end{array}\right.$$  

(2.3)

where $\mathcal{F}(\cdot,v)$ is a $P_\phi$-periodic function for every $v \in \mathbb{R}$, and $\Psi = \psi'$ with $\psi \in \mathcal{H}$.

Hence, there exists a growing mode for the linearized perturbation equations with period $P_\phi$. As for the proof, the main idea hinges on writing this existence problem pertaining to a system of integro-differential equations as an equivalent eigenvalue problem for a fairly complicated operator, which is termed the dispersion operator. Prior to stating the lemma containing this result, we first define the (forward) characteristic curves for the particles in (2.3) since they will appear in the definition of the dispersion operator. Thus, define the curves $X(s,x,v)$ and $V(s,x,v)$, which we shall often abbreviate as $X(s)$ and $V(s)$ respectively, as solutions of the system of ordinary differential equations

$$\left\{\begin{array}{l}
\frac{\partial X}{\partial s} = V(s,x,v), \\
X(0,x,v) = x,
\end{array}\right.$$

$$\left\{\begin{array}{l}
\frac{\partial V}{\partial s} = -\phi'(X(s,x,v)), \\
V(0,x,v) = v.
\end{array}\right.$$  

(2.4)
From the definition of these curves, we can immediately construct an invariant of the physical system, namely the particle energy
\[ e = \frac{1}{2} v^2 + \phi(x) = \frac{1}{2} V(s)^2 + \phi(X(s)) \]
which is independent of the time variable \( s \). This is obtained by multiplying the equation for \( \frac{\partial V}{\partial s} \) in (2.4) by \( V(s) \) and expressing the remaining terms as a total derivative in \( s \). Throughout, we will denote both the particle energy and the exponential function by \( e \), leaving the reader to differentiate between them due to context. For instance, we will often write \( \mu(e) \) in the future instead of \( \mu\left(\frac{1}{2} v^2 + \phi(x)\right) \) or \( \mu\left(\frac{1}{2} V(s)^2 + \phi(X(s))\right) \) even though the three terms are equivalent. With this in hand, we may now define the dispersion operator and state the equivalence result.

**Lemma 2.1.** Let \( \lambda > 0 \) be given. There exists a nontrivial solution to (2.3) of the form
\[ F(t, x, v) = e^{\lambda t} F(x, v) \]
\[ \beta(t, x) = e^{\lambda t} \Psi(x) \]
with \( F \in C^1 \cap L^\infty \) and \( \Psi \in H^1_{\text{per}}(0, P_0) \) if and only if there exists \( \psi \in \mathcal{H} \) such that \( A_\lambda \psi = 0 \). Here, the dispersion operator \( A_\lambda : \mathcal{H} \to \mathcal{L} \) is given by
\[ A_\lambda \psi := -\psi''(x) - \psi(x) \int \mu'(e) dv + \int \mu'(e) \int_{-\infty}^{0} \lambda e^{\lambda s} \psi(X(s)) dsdv \]
where \( X(s) \) is the spatial solution of the characteristic equations (2.4). Additionally, \( F \) and \( \Psi \) are given by
\[ F(x, v) = \mu'(e) \left( \int_{-\infty}^{0} \lambda e^{\lambda s} \psi(X(s)) ds - \psi(x) \right) \]
and \( \Psi = \psi' \).

The ultimate goal, now, is to show that there exists \( \psi \in \mathcal{H} \) and \( \lambda > 0 \) such that \( A_\lambda \psi = 0 \) with \( \psi \neq 0 \), as this will imply the existence of a growing mode and hence the theorem. Thus, we wish to show that this operator has a nontrivial kernel, or stated another way, that zero is an eigenvalue of the operator. Fortunately, we have some information about the eigenvalues of \( A_\lambda \) as the next result shows. For the proof, we refer the reader to [9].

**Lemma 2.2.** For any \( \lambda > 0 \), the operator \( A_\lambda \) is self-adjoint, possesses purely discrete spectrum, and has only real eigenvalues.

In order to show that \( A_\lambda \psi \) may be zero for nonzero \( \psi \), we will demonstrate that it may possess both positive and negative eigenvalues for different values of \( \lambda > 0 \) and utilize the Intermediate Value Theorem to conclude that it must
be zero for some $\lambda$ between these values. This will require continuity of the eigenvalues of $A_\lambda$ with respect to $\lambda$. As such, for every $\lambda \in \mathbb{R}$ define
\begin{equation}
\sigma(\lambda) = \inf_{\psi \in H : \|\psi\|_2 = 1} (A_\lambda \psi, \psi).
\end{equation}
The following result will therefore allow us to consider this approach.

**Lemma 2.3.** The function $\sigma : \mathbb{R} \to \mathbb{R}$ defined by (2.7) is continuous on $(0, \infty)$.

With this in place, we need only display that the eigenvalues of $A_\lambda$ must be positive and negative for different values of $\lambda$. Hence, we first consider what happens as $\lambda \to \infty$.

**Lemma 2.4.** There exists $\Lambda > 0$ such that if $\lambda > \Lambda$, then $A_\lambda$ has only positive eigenvalues and $\sigma(\lambda) > 0$.

Next, we consider $\sigma(\lambda)$ as $\lambda \to 0^+$. However, it does not follow automatically that these values are negative. Instead, we have the following lemma, which decomposes the structure of this limit into more manageable terms.

**Lemma 2.5.** For any $\psi \in H^1_{per}(0, P_\phi)$ we have
\[
\lim_{\lambda \to 0^+} (A_\lambda \psi, \psi) = \int_0^{P_\phi} \left| \psi'(x) \right|^2 - \int \mu'(e) \, dv |\psi(x)|^2 \, dx \\
+ 2 \int_{\phi_+}^{\phi_+} \mu'(e) \frac{1}{P_f} \left( \int_0^{P_\phi} \frac{\psi(y)}{\sqrt{2(e - \phi(y))}} \, dy \right)^2 \, de \\
+ 2 \int_{\phi_-}^{\phi_-} \mu'(e) \frac{1}{P_t} \left( \int_\alpha^{P_\phi-\alpha} \frac{\psi(y)}{\sqrt{2(e - \phi(y))}} \, dy \right)^2 \, de.
\]
Here, $\alpha \in \left[0, \frac{1}{2}P_\phi\right]$ is the unique point in the interval such that $\phi(\alpha) = e$, for a given $e \in [\phi_-, \phi_+]$, while
\[
P_f = \int_0^{P_\phi} \frac{1}{\sqrt{2(e - \phi(y))}} \, dy, \\
P_t = \int_\alpha^{P_\phi-\alpha} \frac{1}{\sqrt{2(e - \phi(y))}} \, dy.
\]

In view of Lemma 2.5, we can now see that the sign of $\mu'(e)$ may play a large role in the stability of solutions. For instance, if $\mu'(e) \leq 0$ for $e \in [\phi_-, \infty)$ one could guarantee the last two expressions remain non-positive while choosing $\psi$ to minimize the first integral. However, as we will show below, BGK waves cannot support such distribution functions, meaning $\mu'(e)$ cannot remain nonpositive for all $e \in [\phi_-, \infty)$.

**Theorem 2.2.** Assume conditions (i)-(iv) hold and the BGK wave $(\mu, \phi)$ satisfies (1.2), then $\int \mu' \left( \frac{1}{2}v^2 + \phi(x) \right) \, dv$ must assume positive values for $x \in [0, P_\phi]$. In particular, no choice of $\mu$ can satisfy $\mu'(e) \leq 0$ for all $e \in [\phi_-, \infty)$. 

Hence, the signs of the complicated terms in the representation of Lemma 2.5 depend very delicately on the behavior of \( \mu'(e) \). Regardless, we can eliminate this difficulty by utilizing the BGK waves and their symmetry to prove Theorem 2.1. Thus, we consider the case of \( \lambda \to 0^+ \) and determine the behavior of \( (A_\lambda \psi, \psi) \) in the case of perturbations with period \( P_\phi \).

**Proof of Theorem 2.1.** Combining Lemmas 2.1, 2.2, 2.3, and 2.4, it suffices to show that

\[
\lim_{\lambda \to 0^+} (A_\lambda \psi, \psi) < 0
\]

for some \( \psi \in H^1_{\text{per}}(0, P_\phi) \) as this will imply a negative eigenvalue of \( A_\lambda \) for \( \lambda > 0 \). In view of Lemma 2.5, we can simplify this statement further. Since the latter terms in the representation are complicated, we will construct a specific choice of \( \psi \in H^1_{\text{per}}(0, P_\phi) \) for which the first piece is negative and use symmetry to eliminate the contribution of the last two expressions. To begin the construction, recall (1.2):

\[
\phi''(x) = 1 - \int \mu(e)dv.
\]

We take an \( x \)-derivative to find

\[
\phi'''(x) = -\left( \int \mu'(e) dv \right) \phi'(x)
\]

or letting \( q(x) = \int \mu'(\frac{1}{2}v^2 + \phi(x)) dv \) as in the introduction, this becomes

\[
\phi'''(x) + q(x)\phi'(x) = 0.
\]

Hence, the function \( u = \phi' \) satisfies the Sturm-Liouville problem

\[
\begin{cases}
  u''(x) + (q(x) + \lambda)u(x) = 0 \\
  u(0) = u(P_\phi) = 0
\end{cases}
\]

for \( u \in H^2(0, P_\phi) \) with corresponding eigenvalue \( \lambda = 0 \). However, \( \phi \) attains a maximum at its half-period and possesses no other critical points on \((0, P_\phi)\). Thus, \( \phi'(\frac{1}{2}P_\phi) = 0 \) and because \( \phi' \) has exactly one root in \((0, P_\phi)\), it must be the second eigenfunction [13]. Therefore, there exists \( u_0 \in \mathcal{H} \) and \( \lambda_0 < 0 \) such that

\[
u_0''(x) + (q(x) + \lambda_0)u_0(x) = 0
\]

and \( u_0(0) = u_0(P_\phi) = 0 \). Using the symmetry of \( q \), namely \( q(P_\phi - x) = q(x) \) for every \( x \in [0, P_\phi] \), we see that the function \( \tilde{u}_0(x) := u_0(P_\phi - x) \) is also a solution of the Sturm-Liouville problem with the same eigenvalue. By uniqueness it follows that there is \( C \in \mathbb{R} \) such that \( u_0(x) = Cu_0(P_\phi - x) \).

Evaluating this at the point \( x = \frac{1}{2}P_\phi \), we see that either \( C = 1 \) or \( u_0(\frac{1}{2}P_\phi) = 0 \). As \( u_0 \) is the first eigenfunction, it has no roots in \((0, P_\phi)\) and thus \( u_0(x) = u_0(P_\phi - x) \). Since \( u_0 \in H^2(0, P_\phi) \) we may take a derivative of (2.8) and find that \( u_0' \) satisfies the symmetry condition \( u_0'(P_\phi - x) = -u_0'(x) \) for every \( x \in [0, P_\phi] \). With these functions in place, define \( \psi(x) = u_0(x)u_0'(x) \) for
x \in [0, P_\phi]. Using the aforementioned properties of u_0 and u_0', we see that \psi satisfies

$$\psi(0) = \psi(P_\phi) = 0$$

with \psi(P_\phi - x) = -\psi(x) for all x \in [0, P_\phi]. In addition, \psi can be extended to the whole space in a smooth manner by imposing \(P_\phi\)-periodicity since

$$\psi'(P_\phi) = |u_0'(P_\phi)|^2 + u_0(P_\phi)u_0''(P_\phi)$$

$$= |u_0'(P_\phi)|^2$$

$$= |u_0'(0)|^2$$

$$= |u_0(0)|^2 + u_0(0)u_0''(0)$$

$$= \psi'(0).$$

As \(u_0 \in H^2(0, P_\phi)\), we see that \(\psi \in H^1(0, P_\phi)\) and since \(\psi\) is smooth at the endpoints, it follows that \(\psi \in H^1_{per}(0, P_\phi)\). Furthermore, this function satisfies

$$|\psi'|^2 - q|\psi|^2 = |u_0'|^2 + u_0u_0'' - q|u_0u_0'|^2$$

$$= |u_0'|^2 - (q + \lambda_0)|u_0|^2 - q|u_0u_0'|^2$$

$$= |u_0'|^4 - 2(q + \lambda_0)|u_0u_0'|^2 + (q + \lambda_0)^2|u_0|^4 - q|u_0u_0'|^2$$

$$= |u_0'|^4 - 2\lambda_0|u_0u_0'|^2 + (q + \lambda_0)^2|u_0|^4 - 3q|u_0u_0'|^2$$

Finally, we compute each portion of the representation given in Lemma 2.5. For the first integral, we have

$$\int_0^{P_\phi} |\psi'(x)|^2 - q(x)|\psi(x)|^2 \, dx$$

$$= \int_0^{P_\phi} |u_0'|^4 - 2\lambda_0|u_0u_0'|^2 + (q + \lambda_0)^2|u_0|^4 - 3q|u_0u_0'|^2 \, dx$$

For the first term of the integral, we integrate by parts and use (2.8) to yield

$$\int_0^{P_\phi} |u_0(x)|^4 \, dx = \int_0^{P_\phi} \left( \frac{d}{dx} u_0(x) \right) (u_0'(x))^3 \, dx$$

$$= -3 \int_0^{P_\phi} u_0(x)|u_0'(x)|^2u_0''(x) \, dx$$

$$= 3 \int_0^{P_\phi} (q(x) + \lambda_0)|u_0(x)u_0'(x)|^2 \, dx$$

Notice here that we have utilized the conditions \(u_0(0) = u_0(P_\phi) = 0\) to eliminate boundary terms arising from integration by parts. Hence, we find

$$\int_0^{P_\phi} |\psi'(x)|^2 - q(x)|\psi(x)|^2 \, dx = \int_0^{P_\phi} \left[ \lambda_0|u_0u_0'|^2 + (q + \lambda_0)^2|u_0|^2 \right] \, dx.$$
The first term of this expression can be simplified further by multiplying 
(2.8) by \( u_0^3 \) and integrating so that

\[
\int_0^{P_\phi} (q(x) + \lambda_0)|u_0(x)|^4 \, dx = - \int_0^{P_\phi} u_0'(x)(u_0(x))^3 \, dx \\
= \int_0^{P_\phi} u_0'(x) \frac{d}{dx} [(u_0(x))^3] \, dx \\
= 3 \int_0^{P_\phi} |u_0'(x)u_0(x)|^2 \, dx.
\]

Using this along with the even symmetry about \( \frac{1}{2}P_\phi \) of \(|u_0|^4\) and \( q \), the first piece of the representation satisfies

\[
\int_0^{P_\phi} \left[ |\psi'(x)|^2 - q(x)|\psi(x)|^2 \right] \, dx = \int_0^{P_\phi} \left[ \frac{1}{3}\lambda_0(q + \lambda_0)|u_0|^4 + (q + \lambda_0)^2|u_0|^4 \right] \, dx \\
= \int_0^{P_\phi} (q + \lambda_0) \left( q + \frac{4}{3}\lambda_0 \right) |u_0|^4 \, dx \\
= 2 \int_0^{P_\phi} (q + \lambda_0) \left( q + \frac{4}{3}\lambda_0 \right) |u_0|^4 \, dx.
\]

Now, since \( \psi \) is odd about \( \frac{1}{2}P_\phi \), the remaining terms in the representation of Lemma 2.5 will vanish. More precisely, for every \( e \in [\phi_+, \infty) \) we have

\[
(2.9) \quad \int_0^{P_\phi} \frac{\psi(y)}{\sqrt{2(e - \phi(y))}} \, dy = 0
\]

and for every \( e \in [\phi_-, \phi_+] \) we have

\[
(2.10) \quad \int_{\alpha}^{P_\phi - \alpha} \frac{\psi(y)}{\sqrt{2(e - \phi(y))}} \, dy = 0.
\]

To prove (2.9), we first decompose the integral into the reflected components of \( \psi \),

\[
\int_0^{P_\phi} \frac{\psi(y)}{\sqrt{2(e - \phi(y))}} \, dy = \int_0^{\frac{1}{2}P_\phi} \frac{\psi(y)}{\sqrt{2(e - \phi(y))}} \, dy + \int_{\frac{1}{2}P_\phi}^{P_\phi} \frac{\psi(y)}{\sqrt{2(e - \phi(y))}} \, dy \\
= I + II.
\]

Now, computing term \( II \), we change variables \( z = P_\phi - y \) and use the symmetry of \( \psi \) and \( \phi \) from (iv) to demonstrate the cancellation of these
terms, so that

\[ II = \int_{P_0}^{P_0} -\psi(P_0 - y) \sqrt{2(e - \phi(y))} \, dy \]

\[ = -\int_{1/2}^{1/2} P_0 \psi(z) \sqrt{2(e - \phi(P_0 - z))} \, dz \]

\[ = -\int_{0}^{1/2} P_0 \psi(z) \sqrt{2(e - \phi(z))} \, dz \]

\[ = -1 \]

Hence, adding these terms yields (2.9). Turning to the justification of (2.10), we again divide the interval of integration about the midpoint \( \frac{1}{2} P_0 \). Using the symmetry and periodicity of \( \phi \) and \( u \), we perform the same change of variables as above to find

\[ \int_{P_0}^{P_0-\alpha} \frac{\psi(y)}{\sqrt{2(e - \phi(y))}} \, dy \]

\[ = \int_{1/2}^{1/2} P_0 \psi(z) \sqrt{2(e - \phi(P_0 - z))} \, dz \]

\[ = -\int_{0}^{1/2} P_0 \psi(z) \sqrt{2(e - \phi(z))} \, dz. \]

Hence, we conclude (2.10) by adding these terms.

Finally, from these computations and in view of assumption (v), we find

\[ \lim_{\lambda \rightarrow 0^+} (A_\lambda \psi, \psi) = \int_{0}^{P_0} \left[ |\psi'(x)|^2 - \int P_0 \mu'(e) \, dv |\psi(x)|^2 \right] \, dx \]

\[ + 2 \int_{\phi_+}^{\phi_+} \mu'(e) \frac{1}{P_1} \left( \int_{0}^{P_0} \frac{\psi(y)}{\sqrt{2(e - \phi(y))}} \, dy \right)^2 \, de \]

\[ + 2 \int_{\phi_-}^{\phi_-} \mu'(e) \frac{1}{P_1} \left( \int_{0}^{P_0-\alpha} \frac{\psi(y)}{\sqrt{2(e - \phi(y))}} \, dy \right)^2 \, de \]

\[ = 2 \int_{0}^{1/2} P_0 \left( q(x) + \lambda_0 \right) \left( q(x) + \frac{4}{3} \lambda_0 \right) |u_0(x)|^4 \, dx \]

\[ < 0. \]

which completes the proof. □

**Remark 2.1.** Though the condition (v) is somewhat difficult to check for a given distribution function, it effectively states that the majority of values of the potential are concentrated between \( -\lambda_0 \) and \( -\frac{4}{3} \lambda_0 \). Of course, this depends upon the values \( \lambda_0 \) may assume and a great deal of study has been
devoted to estimating the first eigenvalue of such a Sturm-Liouville problem. Equivalently, as the second eigenvalue is $\lambda_1 = 0$, one may attempt to determine the length of the gap between the first two eigenvalues. Many authors have investigated questions in this vein under specific conditions on the potential $q$, for instance, see [1] and [8] for results concerning single-well and convex potentials, respectively. In the case of Theorem 2.1, the properties of $q$ greatly depend on the local behavior of the distribution function $\mu$ and potential $\phi$, and even for different single-well or convex potentials, assumption (v) may be satisfied or fail to hold.

We now end the section with the proof of Theorem 2.2.

Proof of Theorem 2.2. Let $\mu \in C^1$ and $\phi \in C^2$ satisfying

$$\phi''(x) = 1 - \int \mu \left( \frac{1}{2} v^2 + \phi(x) \right) \, dv$$

and assumptions (i) - (iv) be given. We will take advantage of the regularity of these functions. Assume

$$\int \mu' \left( \frac{1}{2} v^2 + \phi(x) \right) \, dv \leq 0$$

for all $x \in [0, \frac{1}{2} P_\phi]$. Using the above differential equation, we see that the regularity of the right side implies that $\phi \in C^3$ and

$$\phi'''(x) = -\int \mu' \left( \frac{1}{2} v^2 + \phi(x) \right) \, dv \, \phi'(x).$$

From our assumption on $\mu'$, we see that $\phi'''(x)$ and $\phi'(x)$ must have the same sign for all $x \in [0, \frac{1}{2} P_\phi]$. From (iv), $\phi$ is strictly decreasing on the interval $[0, \frac{1}{2} P_\phi]$ with a maximum and minimum at $\phi(0)$ and $\phi \left( \frac{1}{2} P_\phi \right)$, respectively. Thus, $\phi'(0) = \phi' \left( \frac{1}{2} P_\phi \right) = 0$ and $\phi'(x) < 0$ on the interval $\left( 0, \frac{1}{2} P_\phi \right)$. Therefore, $\phi'' \leq 0$ on $\left( 0, \frac{1}{2} P_\phi \right)$ and thus $\phi''$ is decreasing on $\left( 0, \frac{1}{2} P_\phi \right)$. However, from the above conditions on $\phi'$, we see that $\phi'$ must transition from decreasing to increasing on some subinterval of $\left( 0, \frac{1}{2} P_\phi \right)$, and this implies that $\phi''$ must transition from negative values to positive values on the same subinterval. Clearly such a transition cannot occur if $\phi''(x)$ is decreasing for every $x \in (0, \frac{1}{2} P_\phi)$ and we arrive at a contradiction. Therefore, there exists $x \in \left[ 0, \frac{1}{2} P_\phi \right]$ such that

$$\int \mu' \left( \frac{1}{2} v^2 + \phi(x) \right) \, dv > 0.$$

Additionally, this conclusion cannot be satisfied for any $\mu$ with $\mu'(e) \leq 0$ for all $e \in [\phi_-, \infty)$ and this completes the proof. \qed
3. PROOFS OF LEMMAS 2.1 AND 2.3

In this section, we include proofs of the first two lemmas that were used to show the main results.

**Proof of Lemma 2.1.** Let \( \lambda > 0 \) be given. Since we expect BGK wave solutions to be unstable, we will proceed by searching for solutions to (2.2) of the form,

\[
F(t, x, v) = e^{\lambda t} F(x, v)
\]

\[
\beta(t, x) = e^{\lambda t} \Psi(x).
\]

Here, we consider \( \lambda > 0 \) to capture the possible existence of growing wave modes. Upon dividing by the exponential portion, (2.2) becomes

\[
\begin{align*}
\lambda F + v \partial_x F - \phi' \partial_v F &= v \mu'(e) \Psi \\
\Psi' &= -\int F(x, v) dv.
\end{align*}
\]

We can simplify the first equation using the method of characteristics. Using the previous defined characteristic curves (2.4) we can multiply by the exponential function \( e^{\lambda s} \) and rewrite the first equation in (3.1) as

\[
\frac{d}{ds} \left( e^{\lambda s} F(X(s), V(s)) \right) = e^{\lambda s} \Psi(X(s)) V(s) \mu'(e).
\]

Integrating this equation over the interval \( (-\infty, 0) \), using properties of characteristics, and integrating by parts we find

\[
F(x, v) = \int_{-\infty}^{0} e^{\lambda s} \mu'(e) \Psi(X(s)) V(s) ds
\]

\[
= \int_{-\infty}^{0} e^{\lambda s} \mu'(e) \frac{d}{ds} \psi(X(s)) ds
\]

\[
= \mu'(e) \psi(X(0)) - \int_{-\infty}^{0} \lambda e^{\lambda s} \mu'(e) \psi(X(s)) ds
\]

\[
= \mu'(e) \psi(x) - \mu'(e) \int_{-\infty}^{0} \lambda e^{\lambda s} \psi(X(s)) ds
\]

where the perturbed potential \( \psi \in \mathcal{H} \) satisfies

\[
\psi'(x) = \Psi(x).
\]

Utilizing this representation of \( F \) in the second equation of the system (3.1) yields a single equation for the perturbed potential, namely

\[
\psi''(x) = -\int \mu'(e) \psi(x) dv + \int \mu'(e) \int_{-\infty}^{0} \lambda e^{\lambda s} \psi(X(s)) ds dv.
\]

This can be rewritten as

\[
-\psi''(x) - \psi(x) \int \mu'(e) dv + \int \mu'(e) \int_{-\infty}^{0} \lambda e^{\lambda s} \psi(X(s)) ds dv = 0
\]
and then place this equation in the perspective of an eigenvalue problem for
the operator on the left side. Hence, for every $\lambda \in \mathbb{R}$ we define the dispersion
operator $A_\lambda : \mathcal{H} \to \mathcal{L}$ by

$$
A_\lambda \psi := -\psi''(x) - \psi(x) \int \mu'(e) dv + \int \mu'(e) \int_{-\infty}^{0} \lambda e^{\lambda s} \psi(X(s)) ds dv
$$

and express the above equation as $A_\lambda \psi = 0$. Notice that if $F \in C^1 \cap L^\infty$
and $\Psi \in H^1_{\text{per}}(0, P_\phi)$ are solutions to (2.3), then $\psi \in \mathcal{H}$ since $\psi' = \Psi$.

Conversely, if there is $\psi \in \mathcal{H}$ which satisfies $A_\lambda \psi = 0$, then we can define
$\Psi \in H^1_{\text{per}}$ by $\Psi = \psi'$ and $F \in L^\infty \cap C^1$ by (2.6). Then, we can differentiate
along characteristics as before to recover (2.3).

To conclude this section we prove Lemma 2.3 which guarantees the con-
tinuity of $\sigma$ with respect to $\lambda$.

**Proof of Lemma 2.3.** From (2.7) we have defined

$$
\sigma(\lambda) = \inf_{\{\psi \in \mathcal{H} : ||\psi||_2 = 1\}} (A_\lambda \psi, \psi)
$$

and thus we must measure the differences of these terms with respect to a
change in $\lambda$. Fix $\lambda_0 > 0$ and let $\lambda > 0$ be given. Adding and subtracting
$(\psi, A_\lambda \psi)$ and using (2.5) we find

$$
\sigma(\lambda_0) \leq (\psi, A_\lambda \psi) + (\psi, A_\lambda \psi) - (\psi, A_{\lambda_0} \psi) \\
= (\psi, A_\lambda \psi) + \left| \int_{0}^{P_\psi} \left[ \psi'(x)^2 - \int \mu'(e) dv \psi(x)^2 \right] dx \\
+ \int_{0}^{P_\psi} \int \mu'(e) \psi(x) \int_{-\infty}^{0} \lambda e^{\lambda s} \psi(X(s)) ds dv dx \\
- \int_{0}^{P_\psi} \left[ -\psi'(x)^2 + \int \mu'(e) dv \psi(x)^2 \right] dx \\
- \int_{0}^{P_\psi} \int \mu'(e) \psi(x) \int_{-\infty}^{0} \lambda_0 e^{\lambda_0 s} \psi(X(s)) ds dv dx \right|
$$

$$
\leq (\psi, A_\lambda \psi) + \int_{0}^{P_\psi} \int |\mu'(e)| \psi(x) \int_{-\infty}^{0} |\lambda e^{\lambda s} - \lambda_0 e^{\lambda_0 s}| |\psi(X(s))| ds dv dx
$$
With this we express the innermost portion of the last term as an integral so that the string of inequalities becomes

\[
\sigma(\lambda_0) \leq (\psi, A_\lambda \psi) + \int_0^{P_0} \int_{-\infty}^0 \left| \lambda e^{\lambda s} - \lambda_0 e^{\lambda_0 s} \right| |\psi(x)\mu'(\epsilon)\psi(X(s))| ds d\epsilon dx
\]

\[
= (\psi, A_\lambda \psi) + \int_0^{P_0} \int_{-\infty}^0 \int_{\lambda_0}^\lambda \frac{d}{d\xi} (\xi e^{\xi s}) d\xi |\psi(x)\mu'(\epsilon)\psi(X(s))| ds d\epsilon dx
\]

\[
\leq (\psi, A_\lambda \psi) + \int_0^{P_0} \int_{-\infty}^0 \int_{\lambda_0}^\lambda (\xi|s|e^{\xi s} + e^{\xi s}) d\xi |\psi(x)\mu'(\epsilon)\psi(X(s))| ds d\epsilon dx
\]

Finally, we use \( \|\psi\|_2 = 1 \) and bound this integral in \( \xi \), yielding

\[
\sigma(\lambda_0) \leq (\psi, A_\lambda \psi) + \sup_{x \in [0, P_0]} \int |\mu'(\epsilon)| d\epsilon \int_{\lambda_0}^\lambda \int_{-\infty}^0 (\xi|s|e^{\xi s} + e^{\xi s}) d\xi d\epsilon dx
\]

\[
\leq (\psi, A_\lambda \psi) + \sup_{x \in [0, P_0]} \int |\mu'(\epsilon)| d\epsilon \int_{\lambda_0}^\lambda \frac{2}{\lambda} d\xi
\]

\[
\leq (\psi, A_\lambda \psi) + C |\ln \lambda - \ln \lambda_0|
\]

with \( C \) depending upon \( \sup_{x \in [0, P_0]} \int |\mu'(\epsilon)| d\epsilon \). As \( \psi \in \mathcal{H} \) is arbitrary, we take the infimum over these functions to find

\[
\sigma(\lambda_0) - \sigma(\lambda) \leq C |\ln \lambda - \ln \lambda_0|.
\]

(3.2)

Finally, \( \lambda \) and \( \lambda_0 \) were arbitrary and hence we may switch their involvement in the proof, yielding

\[
\sigma(\lambda) - \sigma(\lambda_0) \leq C |\ln \lambda - \ln \lambda_0|.
\]

Combining this with (3.2), we have

\[
|\sigma(\lambda) - \sigma(\lambda_0)| \leq C |\ln \lambda - \ln \lambda_0|
\]

and the continuity of \( \sigma \) on \((0, \infty)\) follows from the continuity of \( \ln(x) \) on the same interval.
4. Proofs of Lemmas 2.4 and 2.5

Proof of Lemma 2.4. Let $\psi \in \mathcal{H}$ be given. Then, for any $\lambda > 0$ we may use the periodicity of $\psi$ and integration by parts to find

$$
\left( -\psi'', \psi \right) = - \int_0^{P_0} \psi''(x)\psi(x)dx = -\psi'\left|_0^{P_0} \right. + \int_0^{P_0} |\psi'(x)|^2dx
$$

$$
= - \left[ \psi'(P_0)\psi(0) - \psi'(0)\psi(0) \right] + \|\psi'\|^2_2
$$

$$
= \|\psi'\|^2_2.
$$

Since this expression is exactly the first portion of the dispersion operator integrated against $\psi$, we see that a positive part emerges. Thus, we can divide $A_\lambda$ into a clearly positive part plus a complicated portion of the operator which depends upon particle orbits. For convenience we will let

$$
K_\lambda \psi = -\psi(x) \int \mu'(e)dv + \int \mu'(e) \int_{-\infty}^0 \lambda e^{\lambda s} \psi \left( X(s) \right) dsdv
$$

so that from (2.5)

$$
(A_\lambda \psi, \psi) = \|\psi'\|^2_2 + (K_\lambda \psi, \psi).
$$

Now, we can bound $K_\lambda$ as follows. Using (4.1), we integrate by parts in $s$ to find

$$
(K_\lambda \psi, \psi) = \int_0^{P_0} \left[ -\psi(x) \int \mu'(e)dv + \int \mu'(e) \int_{-\infty}^0 \frac{d}{ds} (e^{\lambda s}) \psi \left( X(s) \right) dsdv \right] \psi(x)dx
$$

$$
= \int_0^{P_0} \psi(x) \int \mu'(e) \int_{-\infty}^0 e^{\lambda s} \psi' \left( X(s) \right) V(s) dsdv dx.
$$

Next, let $\alpha = 1 - \frac{3}{4\gamma} - \epsilon$ with $\epsilon > 0$ small enough that $\alpha \in \left( \frac{1}{4\gamma}, \frac{4\gamma-3}{4\gamma} \right)$. Such an $\alpha$ is guaranteed to exist since $\gamma > 1$ in (iii). Then, we use the Cauchy-Schwarz inequality, change variables $(x, v) \mapsto (X(s), V(s))$, and utilize both the measure-preserving property (cf. [3]) of the Hamiltonian system (as the Jacobian is one) and the invariance of the particle energy $e$ to bound $K_\lambda$ so
that
\[
\left| (K\lambda \psi, \psi) \right| \leq \int_0^{P\phi} \int_{-\infty}^{0} \left( |\mu'(e)|^{1-\alpha} e^{\frac{1}{2} \lambda s} |V(s)||\psi'(X(s))| \right) \cdot \\
\left( |\psi(x)||\mu'(e)|^{\alpha} e^{\frac{1}{2} \lambda s} \right) ds dx dv
\]
\[
\leq \left( \int_{-\infty}^{0} \int_0^{P\phi} e^{\lambda s} V(s)^2 \mu'(e)^{2(1-\alpha)} \psi'(X(s))^2 dxdvds \right)^{\frac{1}{2}}.
\]
\[
\leq \left( \int_{-\infty}^{0} \int_0^{P\phi} e^{\lambda s} \mu'(e)^{2(1-\alpha)} \psi(x)^2 dxdvds \right)^{\frac{1}{2}}.
\]
\[
\leq \left[ \left( \int_{-\infty}^{0} e^{\lambda s} ds \right) \left( \sup_{X \in [0, P\phi]} \int V^2 \mu'(e)^{2(1-\alpha)} dV \right) \left( \int_0^{P\phi} \psi'(X)^2 dX \right) \right]^{\frac{1}{2}}.
\]
\[
\leq \frac{1}{\lambda} \|\psi'\|_2 \left( \sup_{X \in [0, P\phi]} \int V^2 \mu'(e)^{2(1-\alpha)} dV \right)^{\frac{1}{2}}.
\]
Using assumption (iii), we have
\[
|\mu'(e)|^{2\alpha} \leq C(1 + |e|)^{-2\alpha \gamma} \leq C(1 + |v|)^{-4\alpha \gamma}
\]
and
\[
V^2 |\mu'(e)|^{2(1-\alpha)} \leq C(1 + |V|)^{2-4(1-\alpha)\gamma}.
\]
Since \(\alpha\) is chosen as above, this implies
\[-4\alpha \gamma < -1 \text{ and } 2 - 4(1 - \alpha)\gamma < -1.\]
Hence, the integrals above are finite and we find
\[
(4.3) \quad \left| (K\lambda \psi, \psi) \right| \leq \frac{C}{\lambda} \|\psi'\|_2 \|\psi\|_2.
\]
Now suppose that \(A\lambda \psi = \eta \psi\) for some \(\eta \neq 0\). Since \(\psi \in \mathcal{H}\), we know that \(\int_0^{P\phi} \psi(x) dx = 0\). Therefore, we can invoke Wirtinger’s inequality (cf. [7]) to find
\[
(4.4) \quad \|\psi\|_2 \leq \frac{P\phi}{2\pi} \|\psi'\|_2.
\]
Finally, putting together (4.2), (4.3), and (4.4) we have
\[ \eta \| \psi \|_2^2 = (\eta \psi, \psi) = (A_\lambda \psi, \psi) = \| \psi' \|_2^2 + (K_\lambda \psi, \psi) \geq \| \psi' \|_2^2 - \frac{C}{\lambda} \| \psi' \|_2 \| \psi \|_2 \geq \frac{C}{\lambda} \| \psi' \|_2 \| \psi \|_2^2 \geq \left( 1 - \frac{CP_\phi}{2\pi\lambda} \right) \| \psi' \|_2^2 \geq 0 \]
for \( \lambda > \Lambda := \frac{CP_\phi}{2\pi} \). Thus, \( \eta > 0 \) as \( \lambda \to \infty \), and the eigenvalues of \( A_\lambda \) are positive for large values of \( \lambda \). 

In order to prove Lemma 2.5 and characterize the behavior of \((A_\lambda \psi, \psi)\) as \( \lambda \to 0^+ \), we will first state a smaller result which will come in handy during the remainder of the section. This lemma was previously proved and used in [9] for the investigation of multi-periodic BGK waves of (VP).

**Lemma 4.1.** For any \( P \)-periodic function \( \theta \in L^1(0,P) \), we have
\[
\lim_{\lambda \to 0^+} \int_{-\infty}^{0} e^s \theta \left( \frac{s}{\lambda} \right) ds = \frac{1}{P} \int_{0}^{P} \theta(s)ds.
\]
We refer the reader to [9] for the proof. Finally, we end the paper with the proof of the final lemma.

**Proof of Lemma 2.5.** We will divide the particles into two groups in order to analyze the complicated term in \( \lim_{\lambda \to 0^+} (A_\lambda \psi, \psi) \). Since the particle energy \( e = \frac{1}{2} v^2 + \phi(x) \) is invariant, we categorize particles by the value of their energy as demonstrated by Figure 1:

1. **Free Particles**, which possess energy \( e > \phi_+ \):
   These particles cannot attain zero velocity as
   \[
   \frac{1}{2} V(s)^2 = e - \phi(X(s)) > \phi_+ - \phi(X(s)) \geq 0.
   \]
   Hence, they cannot stop or change direction within the interval. Therefore, they are free and must travel through the entire interval \([0, P]\) then go back and forth. By the change of variables \( y = X(s) \), we find
   \[
P_f = \int_{0}^{P} ds = \int_{0}^{P_0} \frac{dy}{V(s)}.
   \]
Figure 1. A representative graph of the potential $\phi(x)$ for the BGK waves of (1.1)-(1.2). The particles are divided into the Free Particles, whose energy lies completely above the dashed line ($e > \phi_+$) and hence above the graph of $\phi$, and the Trapped Particles, whose energy lies on or below the dashed line ($\phi_- \leq e \leq \phi_+$) and intersects the graph of $\phi$. For the latter, the symmetry of $\phi$ implies that the dotted line denoting the particle energy must intersect the graph of $\phi$ in exactly two points within the interval $[0, P_\phi]$. One of these points (denoted by $\alpha$) must be less than $\frac{1}{2} P_\phi$ and the other, at the reflected point $P_\phi - \alpha$, must be greater than $\frac{1}{2} P_\phi$.

Thus, the free particles (including those with either strictly positive or strictly negative velocities) will have a half-period of

$$P_f = \int_0^{P_\phi} \frac{1}{\sqrt{2(e - \phi(y))}} dy.$$

(2) Trapped Particles, which possess energy $\phi_- \leq e \leq \phi_+$:

With this constraint on the energy and assumption (iv), we see that $\phi$ will be equal to the particle energy at some point $\alpha \in \left[0, \frac{1}{2} P_\phi\right]$. In fact, by the symmetry of $\phi$ (see Figure 1) this must occur at exactly two such points in the interval $[0, P_\phi]$, namely $\alpha$ and $P_\phi - \alpha$. At these points, the particle velocity must be zero as $|v| = \sqrt{2(e - \phi(x))}$. As $\psi$ is $P_\phi$-periodic, the velocities of such particles change sign and they are trapped in the interval $[\alpha, P_\phi - \alpha]$. The particles then move both back and forth within this interval. Hence, the trapped particles have half-period

$$P_t = \int_\alpha^{P_\phi - \alpha} \frac{1}{\sqrt{2(e - \phi(y))}} dy.$$
Now that we have categorized the particles, we can use this to represent the complicated portion of $\lim_{\lambda \to 0^+} (A_\lambda \psi, \psi)$. Let us denote

$$D(x, v) = \lim_{\lambda \to 0^+} \left( \int_{-\infty}^{0} \lambda e^{s} \psi(X(s, x, v)) \, ds \right).$$

Then, changing variables, invoking Lemma 4.1 for $\theta(s) = \psi(X(s))$, and using the above categorization of the particles, we find

$$D(x, v) = \lim_{\lambda \to 0^+} \int_{-\infty}^{0} e^{s} \psi(X(s)) \, ds$$

$$= \begin{cases} 
\frac{1}{2P_f} \int_{0}^{2P_f} \psi(X(s)) \, ds, & \text{for } e > \phi_+ \\
\frac{1}{2P_f} \int_{0}^{2P_f} \psi(X(s)) \, ds, & \text{for } \phi_- \leq e \leq \phi_+. 
\end{cases}$$

Thus, by changing variables $y = X(s)$ and using $V(s) = \sqrt{2(e - \phi(X(s)))}$, this becomes

$$D(x, v) = \begin{cases} 
\frac{1}{P_f} \int_{0}^{P_\phi} \frac{\psi(y)}{\sqrt{2(e - \phi(y))}} \, dy, & \text{for } e > \phi_+ \\
\frac{1}{P_f} \int_{0}^{P_\phi - \alpha} \frac{\psi(y)}{\sqrt{2(e - \phi(y))}} \, dy, & \text{for } \phi_- \leq e \leq \phi_+, 
\end{cases}$$

and $x \in (\alpha, P_\phi - \alpha)$.

Now, recall from (2.5), the last term of $(A_\lambda \psi, \psi)$ is

$$B(\lambda) := \int_{0}^{P_\phi} \int_{0}^{P_\phi} \mu'(e)\psi(x)\lambda e^{s} \psi(X(s)) \, dsvdx.$$

Since the integrand satisfies

$$\int_{-\infty}^{0} \mu'(e)\psi(x)\lambda e^{s} \psi(X(s)) \, ds \leq ||\psi||_\infty ||\mu'(e)|| \psi(x),$$

for $\lambda > 0$ and this upper bound is integrable in $(x, v)$, we can use Lebesgue’s Dominated Convergence Theorem to pass the limit $\lambda \to 0^+$ inside the $(x, v)$ integrals within $B(\lambda)$. Doing so, changing variables $v \mapsto e$, and using (4.5)
we find
\[
\lim_{\lambda \to 0^+} B(\lambda) = \int_0^P \int_0^{P_0} \frac{\psi(x)}{\sqrt{2(e - \phi(x))}} D(x, \sqrt{2(e - \phi(x))}) dx dv
\]
\[
= 2 \int_{\phi_+}^\infty \mu'(e) \int_0^{P_0} \frac{\psi(x)}{\sqrt{2(e - \phi(x))}} D(x, \sqrt{2(e - \phi(x))}) dx dv \int_0^{P_0} \frac{\psi(y)}{\sqrt{2(e - \phi(y))}} dv |\psi(x)|^2 dx
\]
\[
+ 2 \int_{\phi_+}^{P_0} \mu'(e) \int_0^{P_0-\alpha} \frac{\psi(x)}{\sqrt{2(e - \phi(x))}} D(x, \sqrt{2(e - \phi(x))}) dx dv \int_0^{P_0} \frac{\psi(y)}{\sqrt{2(e - \phi(y))}} dv |\psi(x)|^2 dx
\]
\[
= 2 \int_0^\infty \mu'(e) \frac{1}{P_f} \left( \int_0^{P_0} \frac{\psi(y)}{\sqrt{2(e - \phi(y))}} \right)^2 de
\]
\[
+ 2 \int_{\phi_+}^{P_0} \mu'(e) \frac{1}{P_f} \left( \int_0^{P_0-\alpha} \frac{\psi(y)}{\sqrt{2(e - \phi(y))}} \right)^2 de.
\]

Finally, we use this representation in (2.5) and integrate by parts to find
\[
\lim_{\lambda \to 0^+} (A_\lambda \psi, \psi) = \lim_{\lambda \to 0^+} \left( - \int_0^{P_0} \psi(x) \psi''(x) dx + \int_0^{P_0} \frac{\psi(x)}{\sqrt{2(e - \phi(x))}} D(x, \sqrt{2(e - \phi(x))}) dx dv + B(\lambda) \right)
\]
\[
= \lim_{\lambda \to 0^+} \left( \int_0^{P_0} \left[ |\psi'(x)|^2 - \int_0^{P_0} \frac{\psi(x)}{\sqrt{2(e - \phi(x))}} D(x, \sqrt{2(e - \phi(x))}) dx dv \right] dx + B(\lambda) \right)
\]
\[
= \int_0^{P_0} \left[ |\psi'(x)|^2 - \int_0^{P_0} \frac{\psi(x)}{\sqrt{2(e - \phi(x))}} D(x, \sqrt{2(e - \phi(x))}) dx dv \right] dx + \lim_{\lambda \to 0^+} B(\lambda)
\]
and the result follows. \qed

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References


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