MATH 334 REVIEW PROBLEMS

1. A computer password consists of eight characters.
   
   (a) How many different passwords are possible if each character may be any lower case letter or digit?
   
   (b) How many different passwords are possible if each character may be any lower case letter or digit, and at least one character must be a digit?
   
   (c) A computer system requires that passwords contain at least one digit. If eight characters are generated at random, and each is equally likely to be any of the 26 letters or 10 digits, what is the probability that a valid password will be generated?

2. Ten people sit around a circular table in random order. Two of the people are Herman and Gertrude. What is the probability that they sit next to each other?

3. The probability that a set of twins is identical is \( p \). Identical twins are always of the same sex, and the probability that both are boys is 0.5. The sexes of fraternal twins are independent, with each twin having probability 0.5 to be a boy. A set of twins is chosen at random. Find \( P(\text{both are girls}) \) in terms of \( p \).

4. Suppose the IQs of a million people have mean 100 and standard deviation 10.
   
   (a) Find an upper bound on the proportion of scores that exceed 130.
   
   (b) Assume the distribution of scores is symmetric around 100. Find a sharper upper bound on the proportion of scores that exceed 130.

5. Grandma is trying out a new recipe for raisin bread. Each batch of bread dough makes three loaves, and each loaf contains 20 slices of bread.
   
   (a) If she puts 200 raisins into a batch of dough, what is the probability that a randomly chosen slice of bread contains 5 raisins?
   
   (b) How many raisins must she put in so that the probability that a randomly chosen slice will have no raisins is 0.01?

6. Suppose that A and B take turns tossing a biased coin that comes up heads with probability \( p \). Suppose that A tosses first. Find the probability that A tosses the first head.

7. Three cards are dealt off the top of a deck. What is the probability that the first card is an ace given that the third card is an ace?

8. A quality control program at a plastic bottle production line involves inspecting finished bottles for flaws such as microscopic holes. The proportion of bottles that actually have such a flaw is only 0.0002. If a bottle has a flaw, the probability is 0.995 that it will fail the inspection. If a bottle does not have a flaw, the probability is 0.99 that it will pass the inspection.
   
   (a) If a bottle fails inspection, what is the probability that it has a flaw?
   
   (b) If a bottle passes inspection, what is the probability that it does not have a flaw?

9. A traffic light at a certain intersection is green 50% of the time, yellow 10% of the time, and red 40% of the time. A car approaches this intersection once each day. Let \( X \) represent the number of days that pass up to and including the first time the car encounters a red light. Assume that each day represents an independent trial.
   
   (a) Find \( P(X = 3) \)  
   
   (b) Find \( P(X \leq 3) \)  
   
   (c) Find \( E(X) \)  
   
   (d) Find \( V(X) \)

10. Refer to Exercise 9. Let \( Y \) denote the number of days up to and including the third day on which a red light is encountered.
    
    (a) Find \( P(Y = 7) \).  
    
    (b) Find \( E(Y) \).  
    
    (c) Find \( V(Y) \).

11. Refer to Exercise 9. Let \( Z \) denote the number of days in a five-day week on which a red light is encountered.
    
    (a) Find \( P(Z = 2) \).  
    
    (b) Find \( E(Z) \).  
    
    (c) Find \( V(Z) \).
12. Assume that the probability that it snows in Denver is 0.3, the probability that it snows in Golden is 0.4, and the probability that it snows in both Denver and Golden is 0.2. Find the probability that it snows in neither Denver nor Golden.

13. An urn contains 3 red balls and 5 green balls. A fair coin is tossed. If it comes up heads, two balls are selected without replacement from the urn. If it comes up tails, three balls are selected without replacement. Find the probability that all the balls selected are red.

14. Haemophilia is a sex-linked genetic disease that results in the inability of blood to clot. (A disease is sex-linked if the disease gene is located on the X-chromosome.) A woman with one copy of the gene is a carrier, which means that she does not have the disease, but she can transmit it to her male children. If a carrier has children by a man who is disease-free, each son has probability 0.5 of having the disease, and each daughter has probability 0.5 of being a carrier.

(a) A woman whose mother was a carrier has probability 0.5 of being a carrier. If this woman has a son by a man who is disease free, what is the probability that the son has the disease?

(b) If the son does not have the disease, what is the probability that his mother is a carrier?

15. Bottles are labeled as containing 12 ounces of liquid. The process that fills the bottles produces a volume with standard deviation 0.02 ounces and whose mean can be set through calibration. Assume the fill volumes are normally distributed. To what value should the mean be set so that no more than 1% of the bottles will be underfilled?

16. A radioactive mass emits particles at a mean rate of 4 per minute.

(a) What is the probability that exactly three particles are emitted in a 30-second time period?

(b) What is the probability that more than 15 seconds elapse before the first particle is emitted?

(c) If 15 seconds elapse with no particle emitted, what is the probability that no particle is emitted in the next 15 seconds?

17. A meteorologist in Denver recorded the number of days of rain during a 30-day period, and called it $Y$. Does $Y$ have a binomial distribution? Explain.

18. A telephone survey is performed by calling numbers at random until someone answers. Let $Y$ be the number of calls made before the first person answers. Does $Y$ have a binomial distribution? Explain.

19. A new surgical procedure is successful with probability $p$. Assume the procedure is performed five times, and the results are independent of one another.

(a) Find the probability that all five procedures are successful if $p = 0.8$.

(b) Find the probability that exactly four procedures are successful if $p = 0.6$.

(c) Find the probability that fewer than two procedures are successful if $p = 0.3$.

20. A missile protection system consists of $n$ radar sets operating independently, each with probability 0.9 of detecting a missile entering a zone.

(a) If $n = 5$ and a missile enters the zone, what is the probability that exactly four radar sets detect it?

(b) How large must $n$ be so that the probability that a missile is detected by at least one radar set is 0.9999?

21. An oil exploration firm is formed with enough capital to finance ten explorations. The probability that any particular exploration is successful is 0.1. Assume the explorations are independent. Find the mean and variance of the number of successful explorations.

22. Ten percent of the item in a very large lot are defective. Four items are selected at random. Let $Y$ be the number of defective items selected.

(a) Find the expected number of defective items.

(b) The defective items will be repaired, and the cost of repairing $Y$ items is $C = 3Y^2 + Y + 2$. Find the expected cost.
23. An oil prospector will drill a succession of holes in a effort to find a productive well. The probability of success on any given trial is 0.2 and the trials are independent.

(a) What is the probability that the third hole drilled is the first to yield a productive well?
(b) If the prospector can afford to drill at most ten wells, what is the probability that he fails to find a productive well?

24. Two people took turns tossing a fair die until one of them tossed a 6. Person A tossed first. Given that person B threw the first 6, what is the probability that B obtained the the first 6 on her second toss (the fourth toss overall)?

25. The probability that you get a busy signal when calling a certain office is 0.6. Calls are independent.

(a) What is the mean number of calls up to and including the first call that gets through?
(b) You and your friend are both calling the office. What is the probability that it takes exactly four tries in total before both of you get through?

26. There are 30 restaurants in a certain town. Assume that four of them have health code violations. A health inspector chooses 10 restaurants at random to inspect.

(a) What is the probability that two of the restaurants with violations will be visited?
(b) What is the probability that fewer that two of the restaurants with violations will be visited?

27. A company buys a policy to insure its revenue in the event of major snowstorms that shut down business. The policy pays nothing for the first such snowstorm of the year and $10,000 for each one thereafter, until the end of the year. The number of major snowstorms per year that shut down business is assumed to have a Poisson distribution with mean 1.5. Calculate the expected amount paid to the company under this policy during a one-year period.

28. Cars arrive at a toll booth according to a Poisson process with mean 80 cars per hour. What is the probability that at least one car arrives in a one-minute period?

29. A parking lot has two entrances. Cars arrive at entrance I according to a Poisson process with mean three per hour and at entrance II according to a Poisson process with mean four per hour. The numbers of cars arriving at the two entrances are independent. What is the probability that a total of three cars will arrive at the parking lot in a given hour?
MATH 334 REVIEW SOLUTIONS

1. There are 26 letters and 10 digits.
   (a) \(36^8 = 2.82 \times 10^{12}\)
   (b) The number of passwords of letters and digits is \(36^8\). The number of passwords that are all letters is \(26^8\). Therefore the number that contain at least one digit is \(36^8 - 26^8 = 2.61 \times 10^{12}\).
   (c) \(\frac{36^8 - 26^8}{36^8} = 0.926\)

2. The number of combinations of two seats chosen from 10 is \(\frac{10!}{8!2!} = 45\). Of these, 10 are adjacent pairs. Therefore the probability is \(\frac{10}{45}\).

3. Let \(I\) denote the event that the twins are identical and let \(F\) denote the event that the twins are fraternal, and let \(GG\) denote the event that both are girls.
   Now \(P(GG) = P(GG | I)P(I) + P(GG | F)P(F) = 0.5p + 0.25(1 - p)\).

4. Let \(X\) be the IQ score of a randomly chosen person.
   (a) \(P(X > 130) \leq P(|X - 100| > 30) \leq \frac{1}{9}\) by Chebyshev’s inequality.
   (b) Since the distribution is symmetric around 100, we know that \(P(X > 130) = P(X < 70)\), or equivalently, that \(P(|X - 100| > 30) = 2P(X > 130)\).
   Now \(P(|X - 100| > 30) \leq \frac{1}{9}\) by Chebyshev’s inequality. Therefore \(P(X > 130) \leq \frac{1}{18}\).

5. Let \(X\) be the number of raisins in a randomly chosen slice.
   (a) Since the mean number of raisins per slice is \(\frac{200}{60} = \frac{10}{3}\), \(X \sim \text{Poisson}(\frac{10}{3})\). Therefore \(P(X = 5) = e^{-\frac{10}{3}} \left(\frac{10}{3}\right)^5 \frac{1}{5!} = 0.1223\).
   (b) Let \(\lambda\) be the required mean number of raisins per slice. Then \(X \sim \text{Poisson}(\lambda)\), so \(P(X = 0) = e^{-\lambda} = 0.01\). It follows that \(\lambda = \log 100\), so the total number of raisins needed is \(60 \log(100) \approx 276\).

6. Method 1. The Mildly Exciting Way:
   
   \[ P(A \text{ tosses the first head}) = \sum_{n=0}^{\infty} P(A \text{ tosses the first head on the } 2n + 1 \text{th toss}) = \sum_{n=0}^{\infty} p(1 - p)^{2n} = p \sum_{n=0}^{\infty} [(1 - p)^2]^n = \frac{p}{1 - (1 - p)^2} = \frac{1}{2 - p}. \]

   Method 2. The Very Exciting Way:
   Let \(A\) denote the event that \(A\) tosses the first head and let \(B\) denote the event that \(B\) tosses the first head. Now \(B\) cannot toss the first head unless \(A\)'s first toss is a tail. Therefore
   \[ P(B) = P(B \cap A's \text{ first toss is a tail}) = P(A's \text{ first toss is a tail})P(B | A's \text{ first toss is a tail}) = (1 - p)P(B | A's \text{ first toss is a tail}). \]
   Now here comes the exciting part. If \(A's \text{ first toss is a tail}, \text{then it's just as if the game were starting over with } B \text{ tossing first. Therefore } P(B | A's \text{ first toss is a tail}) = P(A). \]
   We now have \(P(B) = (1 - p)P(A)\). Since \(P(B) = 1 - P(A)\), it follows that \(P(A) = \frac{1}{2 - p}\).

7. Method 1. The Hard Way: Let \(A\) denote the event that a card is an ace, let \(N\) denote the event that a card is not an ace, and let \(x\) denote the event that a card is any card. Then
\[ P(A | xA) = \frac{P(A \cap xA)}{P(xA)} = \frac{P(Ax)}{P(xA)}. \] Now \( P(xA) = 4/52, \) and \( P(Ax) = P(AN) + P(AA) = 2/52 \cdot 52 = 2/52. \] Therefore \( P(A | xA) = \frac{4 \cdot 3}{52} \cdot 52 = \frac{3}{51}. \)

**Method 2. The Easy Way:** If the third card is an ace, then there are three aces left among the remaining 51 cards. The probability that the first card is an ace is 3/51.

8. Let \( F \) denote the event that a bottle has a flaw, and let \( I \) be the event that the bottle passes inspection. Then \( P(F) = 0.0002, P(I \mid F)0.005, \) and \( P(I \mid F^c) = 0.99. \)
   \[
   \begin{align*}
   (a) \quad P(F \mid I^c) &= \frac{P(I^c \mid F)P(F)}{P(I^c \mid F)P(F) + P(I^c \mid F^c)P(F^c)} = \frac{(0.995)(0.0002)}{(0.995)(0.0002) + (0.01)(0.9998)} = 0.0195. \\
   (b) \quad P(F^c \mid I) &= \frac{P(I \mid F^c)P(F^c)}{P(I \mid F^c)P(F^c) + P(I \mid F)P(F)} = \frac{(0.99)(0.9998)}{(0.99)(0.9998) + (0.005)(0.0002)} = 0.9999899. 
   \end{align*}
   \]

9. \( X \sim \text{Geom}(0.4). \)
   \[
   \begin{align*}
   (a) \quad P(X = 3) &= (0.4)(0.6)^2 = 0.144 \\
   (b) \quad P(X \leq 3) &= 1 - P(X > 3) = 1 - (0.6)^3 = 0.784 \\
   (c) \quad E(X) &= 1/0.4 = 2.5 \\
   (d) \quad V(X) &= 0.6/(0.4)^2 = 3.75 
   \end{align*}
   \]

10. \( Y \sim \text{NB}(3, 0.4). \)
    \[
    \begin{align*}
    (a) \quad P(Y = 7) &= \frac{6!}{2!(4)!}(0.4)^3(0.6)^4 = 0.124416 \\
    (b) \quad E(Y) &= 3/0.4 = 7.5 \\
    (c) \quad V(Y) &= 3(0.6)/(0.4)^2 = 11.25 \n    \end{align*}
    \]

11. \( Z \sim \text{Bin}(5, 0.4). \)
    \[
    \begin{align*}
    (a) \quad P(Z = 2) &= \frac{5!}{2!(3)!}(0.4)^2(0.6)^3 = 0.3456 \\
    (b) \quad E(Z) &= 5(0.4) = 2 \\
    (c) \quad V(Z) &= 5(0.4)(0.6) = 1.2 
    \end{align*}
    \]

12. Let \( D \) represent the event that it snows in Denver and let \( G \) represent the event that it snows in Golden. Then \( P(D^c \cap G^c) = P[(D \cup G)^c] = 1 - P(D \cup G) = 1 - P(D) - P(G) + P(D \cap G) = 1 - 0.3 - 0.4 + 0.2 = 0.5. \)

13. Let \( R \) represent the event that all the balls are red. Then \( P(R) = P(R \mid H)P(H) + P(R \mid T)P(T) = (3/8)(2/7)(1/2) + (3/8)(2/7)(1/6)(1/2) = 1/16. \)

14. (a) Let \( D \) represent the event that the son has the disease and let \( C \) represent the event that the daughter is a carrier. Then \( P(D) = P(D \mid C)P(C) + P(D \mid C^c)P(C^c) = (0.5)(0.5) + (0)(0.5) = 0.25 \)
   \[
   \begin{align*}
   (b) \quad P(C \mid D^c) &= \frac{P(D^c \mid C)P(C)}{P(D^c \mid C)P(C) + P(D^c \mid C^c)P(C^c)}.
   \end{align*}
   \]
    Now \( P(D^c \mid C) = 0.5, P(C) = 0.5, P(D^c \mid C^c) = 1, P(C^c) = 0.5. \) Therefore \( P(C \mid D^c) = 1/3. \)
15. Let $X$ be the volume of a randomly chosen bottle. We must find $\mu$ so that $P(X < 12) = 0.01$. Let $Z = \frac{X - \mu}{0.02}$. Then $P(X < 12) = P \left( Z < \frac{12 - \mu}{0.02} \right)$. Because $Z \sim N(0,1)$, $P(Z < -2.33) = 0.01$. Therefore $\frac{12 - \mu}{0.02} = -2.33$, so $\mu = 12.0466$.

16. (a) Let $X$ be the number of particles emitted in 30 seconds. Then $X \sim \text{Poisson}(2)$, so

$$P(X = 3) = e^{-2} \frac{2^3}{3!} = 0.1804.$$  

(b) Let $T$ be the time in minutes at which the first particle is emitted. Then $T \sim \text{Exp}(4)$, so

$$P(T > 1/4) = e^{-4(1/4)} = 0.3679.$$  

(c) By the lack of memory property. $P(X > 1/2 | X > 1/4) = P(X > 1/4) = 0.3679$.

17. No, because the days are not independent.

18. No. The number of trials up to the first success has a geometric distribution, not a binomial distribution.

19. (a) Let $X$ be the number that are successful. Then $X \sim \text{Bin}(5, p)$.

$$P(X = 5) = \binom{5}{5} p^5 (1-p)^0 = 0.3277.$$  

(b) $P(X = 4) = \frac{5!}{4!} (0.6)^4 (0.4)^1 = 0.2592$.

(c) $P(X < 2) = P(X = 0) + P(X = 1) = \binom{5}{0} p^0 (1-p)^5 + \binom{5}{1} p^1 (1-p)^4 = 0.5282$.

20. (a) Let $X$ be the number of sets that detect. Then $X \sim \text{Bin}(5, 0.9)$. $P(X = 4) = 0.328$.

(b) $P(X > 0) = 1 - P(X = 0) = 1 - 0.00001 = 0.9999$.

(c) $P(X > 0) = 0.999$, so $P(X = 0) = 0.1^n = 0.001$. So $n = 3$.

21. Let $X$ be the number of successful explorations. Then $X \sim \text{Bin}(10, 0.1)$. $E(X) = 10(0.1) = 1$, and $V(X) = 10(0.1)(0.9) = 0.9$.

22. (a) $Y \sim \text{Bin}(4, 0.1)$, so $E(Y) = 0.4$.

(b) $V(Y) = 0.36$. Now $E(Y^2) = V(Y) + E(Y)^2 = 0.36 + 0.4^2 = 0.52$.

So $E(C) = 3E(Y^2) + E(Y) + 2 = 3.96$.

23. (a) $(0.8)(0.8)(0.2) = 0.128$.

(b) $(0.8)^{10} = 0.1074$.

24. Let $X$ be the number of rolls up to and including the first six. If B threw the first six then $X$ is even.

$$P(X = 4 | X \text{ is even}) = \frac{P(X = 4)}{P(X = 2 \cup X = 4 \cup \cdots)} = \frac{(1/6)(5/6)^3}{\sum_{n=1}^{\infty} (1/6)(5/6)^{2n-1}}.$$  

Now $\sum_{n=1}^{\infty} (1/6)(5/6)^{2n-1} = (1/6)(6/5) \sum_{n=1}^{\infty} (25/36)^n = (1/5) \frac{25/36}{1 - 25/36} = (1/5)(25/11) = 5/11 = 0.4545$.

So $P(X = 4 | X \text{ is even}) = \frac{P(X = 4)}{P(X = 2 \cup X = 4 \cup \cdots)} = \frac{(1/6)(5/6)^3}{\sum_{n=1}^{\infty} (1/6)(5/6)^{2n-1}} = 0.2122$. 

25. (a) Let \( X \) be the number of calls up to and including the first one that gets through.
   Then \( X \sim \text{Geom}(0.4) \). \( E(X) = 1/0.4 = 2.5 \).

   (b) Let \( X \) be the number of calls up to and including the second success. Then \( X \) is has the negative binomial distribution with parameters 2 and 0.4. \( P(X = 4) = \binom{4}{3}0.4^30.6^1 = 0.1728 \)

26. (a) Let \( Y \) be the number of restaurants with health code violations that are visited.
   \[ P(Y = 2) = \frac{\binom{26}{4}}{\binom{30}{10}} = 0.3120. \]

   (b) \[ P(Y < 2) = P(Y = 0) + P(Y = 1) = \frac{\binom{26}{0}}{\binom{30}{10}} + \frac{\binom{26}{4}}{\binom{30}{10}} = 0.1768 + 0.4160 = 0.5928. \]

27. Let \( X \) be the number of snowstorms and let \( Y \) be the amount paid. Then \( Y = 0 \) if \( X = 0 \), and \( Y = 10000(X - 1) \) if \( X > 0 \).
   \[ E(Y) = 0P(X = 0) + \sum_{x=1}^{\infty} 10000(x-1)e^{1.5} \frac{1.5^x}{x!} \]
   \[ = 10000 \sum_{x=1}^{\infty} xe^{-1.5} \frac{1.5^x}{x!} - 10000 \sum_{x=1}^{\infty} e^{-1.5} \frac{1.5^x}{x!} \]
   \[ = 10000[E(X) - P(X > 0)] = 10000(1.5 - 1 - e^{-1.5}) = 7231 \]

28. Let \( X \) be the number of cars that arrive in one minute.
   Then \( X \sim \text{Poisson}(4/3) \). \( P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-4/3} = 0.7364. \)

29. Let \( X \) be the number of cars arriving at entrance I and let \( Y \) be the number of cars arriving at entrance II. Then
   \[ P(X + Y = 3) = P(X = 3 \cap Y = 0) + P(X = 2 \cap Y = 1) + P(X = 1 \cap Y = 2) + P(X = 0 \cap Y = 3) \]
   \[ = P(X = 3)P(Y = 0) + P(X = 2)P(Y = 1) + P(X = 1)P(Y = 2) + P(X = 0)P(Y = 3) \]
   \[ = e^{-3} \frac{3^3}{3!} e^{-4} \frac{4^0}{0!} + e^{-3} \frac{3^2}{2!} e^{-4} \frac{4^1}{1!} + e^{-3} \frac{3^1}{1!} e^{-4} \frac{4^2}{2!} + e^{-3} \frac{3^0}{0!} e^{-4} \frac{4^3}{3!} = 0.0521 \]