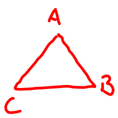
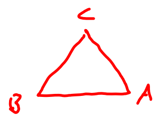
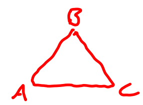
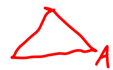
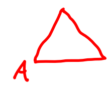
General Relativity HW2 Problems

1. Inspired by our example of the group of 2D rotations that carry the corners of a square into corners, consider the set of 2D rotations that carry the corners of an equilateral triangle into themselves. Develop a three-dimensional faithful representation of this group and list both the “vectors” that correspond to the states, as well as the matrix transformations between them.

Consider:  $= \begin{pmatrix}1\\0\\0\end{pmatrix}$,  $= \begin{pmatrix}0\\1\\0\end{pmatrix}$  $= \begin{pmatrix}0\\0\\1\end{pmatrix}$

Actually we could just think of:  $= \begin{pmatrix}1\\0\\0\end{pmatrix}$  $= \begin{pmatrix}0\\1\\0\end{pmatrix}$  $= \begin{pmatrix}0\\0\\1\end{pmatrix}$

Either way we need to find the elements of $G = \{ I, R(120°), R(240°) \}$

$I = \begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}$ and of course $I\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $I\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $I\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

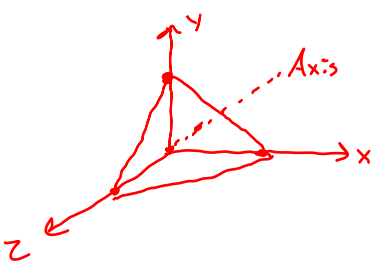
To find $R(120°)$ we need: $R(120°)\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $R(120°)\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$, $R(120°)\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$

After some inspection we find: $R(120°) = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$ works!

Lastly we know $R(240°) = R(120°)R(120°) = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix} = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$

and just checking: $R(240°)\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$, $R(240°)\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $R(240°)\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ ✓

To visualize this representation we can consider the triangle with each vertex on a coordinate axis in 3D. The 2D rotations we are doing correspond to spinning this triangle about the axis shown (which comes out "evenly" between all of the coordinate axes). Note that we could have started with this picture, but then getting the matrices would have been more challenging.



2. Show explicitly that the transformation matrix $\Lambda=\begin{pmatrix} \gamma & -\gamma\frac{v}{c}cos\phi & -\gamma\frac{v}{c}sin\phi & 0 \\ -\gamma\frac{v}{c} & \gamma cos\phi & \gamma sin\phi & 0 \\ 0 & -sin\phi & cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ where $\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ satisfies $\Lambda^T\eta\Lambda=\eta$. Describe what this transformation does in words.

$$\Lambda^T \eta \Lambda = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c}\cos\phi & \gamma\cos\phi & -\sin\phi & 0 \\ -\gamma\frac{v}{c}\sin\phi & \gamma\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\frac{v}{c}\cos\phi & -\gamma\frac{v}{c}\sin\phi & 0 \\ -\gamma\frac{v}{c} & \gamma\cos\phi & \gamma\sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\text{I will do these first}}$$

$$\begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c}\cos\phi & \gamma\cos\phi & -\sin\phi & 0 \\ -\gamma\frac{v}{c}\sin\phi & \gamma\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\gamma & \gamma\frac{v}{c}\cos\phi & \gamma\frac{v}{c}\sin\phi & 0 \\ -\gamma\frac{v}{c} & \gamma\cos\phi & \gamma\sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Observe: $\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

$$\Rightarrow \gamma^2-\gamma^2\frac{v^2}{c^2}=\gamma^2\left(1-\frac{v^2}{c^2}\right) = \frac{1-\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} = 1$$

Top row: $-\gamma^2+\gamma^2\frac{v^2}{c^2}$ $\quad \gamma^2\frac{v}{c}\cos\phi-\gamma^2\frac{v}{c}\cos\phi$ $\quad \gamma^2\frac{v}{c}\sin\phi-\gamma^2\frac{v}{c}\sin\phi$ $\quad 0$ $\quad \Rightarrow (-1\ 0\ 0\ 0)$

2nd row: $\gamma^2\frac{v}{c}\cos\phi-\gamma^2\frac{v}{c}\cos\phi$ $\quad -\gamma^2\frac{v^2}{c^2}\cos^2\phi+\gamma^2\cos^2\phi+\sin^2\phi$ $\quad -\gamma^2\frac{v^2}{c^2}\cos\phi\sin\phi+\gamma^2\cos\phi\sin\phi-\cos\phi\sin\phi$ $\quad 0$

0 $\quad \left(-\gamma^2\frac{v^2}{c^2}+\gamma^2\right)\cos^2\phi+\sin^2\phi$ $\quad \left(-\gamma^2\frac{v^2}{c^2}+\gamma^2\right)\cos\phi\sin\phi-\cos\phi\sin\phi$ $\quad 0$

0 $\quad 1\cos^2\phi+\sin^2\phi$ $\quad 1\cos\phi\sin\phi-\cos\phi\sin\phi$ $\quad 0$ $\quad \Rightarrow (0\ 1\ 0\ 0)$

3rd row: $\gamma^2\frac{v}{c}\sin\phi-\gamma^2\frac{v}{c}\sin\phi$ $\quad -\gamma^2\frac{v^2}{c^2}\sin\phi\cos\phi+\gamma^2\sin\phi\cos\phi-\sin\phi\cos\phi$ $\quad -\gamma^2\frac{v^2}{c^2}\sin^2\phi+\gamma^2\sin^2\phi+\cos^2\phi$ $\quad 0$

0 $\quad \left(-\gamma^2\frac{v^2}{c^2}+\gamma^2\right)\sin\phi\cos\phi-\sin\phi\cos\phi$ $\quad \left(-\gamma^2\frac{v^2}{c^2}+\gamma^2\right)\sin^2\phi+\cos^2\phi$ $\quad 0$ $\quad \Rightarrow (0\ 0\ 1\ 0)$

4th row: $(0\ 0\ 0\ 1)$

Altogether: $\Lambda^T\eta\Lambda = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \eta$ ✓

To figure out what transformations we are dealing with let's simplify:

$$\Lambda(\phi=0) = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which we recognize as a boost along x by v (call this Λ_{tx})

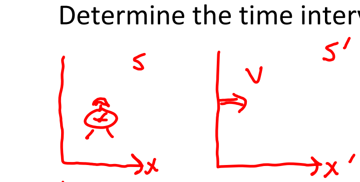
$$\Lambda(v=0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is a rotation in x-y by ϕ (call this Λ_{xy})

But order is important, since $\Lambda_{tx}\Lambda_{xy} \neq \Lambda_{xy}\Lambda_{tx}$. Doing the multiplication we find the given Λ is obtained from $\Lambda = \Lambda_{tx}\Lambda_{xy}$, but when we act on a 4-vector w/ this, i.e. ΛV, we are actually acting w/ Λ_{xy} first, then Λ_{tx}. So this is a rotation by ϕ in x-y followed by a boost w/ v along x.

Now that you have some familiarity with the Lorentz transformations you can use these to derive some classic results in special relativity. The following results can be derived using a simple boost transformation along the *x* axis. I know these are derived in many different texts, but try to do the following problems using what I give you in the problem itself and the explicit form of the boost transformation. My aim is to have you realize that from the perspective of 4D Lorentz transformations most of Special Relativity is quite straightforward even if counter-intuitive from the 3D perspective.

3. "Time-Dilation" A frame S' is moving with respect to frame S with a speed v along $+x$. Consider two events which in S have zero spatial separation, i.e. $\Delta x = \Delta y = \Delta z = 0$, but some nonzero Δt. These could be two ticks of a clock which is at rest in S, so we can call this Δt_{rest}. Note that in S' the clock will be moving along $-x$ with speed v.
Determine the time interval between the two events as measured in S', i.e. $\Delta t'$.



Clock is at rest in S so $\Delta x = \Delta y = \Delta z = 0$ and $\Delta t = \Delta t_{rest}$

In S $\begin{pmatrix} c\Delta t \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} c\Delta t \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma c\Delta t \\ -\beta\gamma c\Delta t \\ 0 \\ 0 \end{pmatrix}$

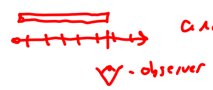
So in S' $\Delta t' = \gamma \Delta t = \gamma \Delta t_{rest}$ since $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \geq 1$

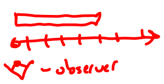
moving clock has larger time-intervals so time is dilated.

It is important to note that all measurements can be made locally. In S we can stand right next to the clock and listen to its ticks. In S' we can hear the first tick as the clock passes our position and then arrange for the clock to send a light pulse to us when it experiences its second tick. Using the time that we receive the second pulse, combined with the known velocity of the clock and combined with c, we can calculate the time between pulses in S' using only local information.

4. (Bonus Problem not to be covered on quiz) "Length-Contraction" A frame S' is moving with respect to frame S with a speed v along $+x$. Consider an object along x' which is at rest with respect to S', e.g. $|\Delta x'| = L_{rest}$. From the perspective of frame S, the object is moving along the x axis with a speed v and we an observer in S can make a measurement of its length by recording the time that the one end passes the origin and then the time when the other end passes the origin. This will yield a value Δt, i.e. a time interval in S. Using Δt and v an observer in S would calculate that $L = v\Delta t$. Thus the two events in S would have the coordinate separations $\Delta t = L/v$, $\Delta x = \Delta y = \Delta z = 0$. Use this to determine the corresponding length in the frame S', i.e. $L = \Delta x'$.

This one is a bit trickier because the quantity we are after (the spatial seperation between the ends at an instant) is not something that us poor sub-luminal humans have easy access to.

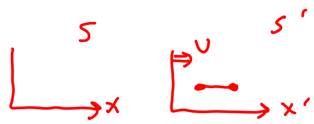
Consider how we would measure the length in the rest frame of the stick. We would lay it down next to a ruler  and then stand at one end and read it 

and then move to the other end and read it 

. Note that moving to the other end takes time ($\Delta t \neq 0$) but that is okay since the stick is at rest so nothing is changing.

Everything gets much trickier if the stick is moving. If we tried to use a ruler, we would still have to move from one end to the other (to make local observations) but in this $\Delta t \neq 0$ the stick has moved!

A better method is to look at one position $\Delta x = \Delta y = \Delta z = 0$ and time how long between when the first end of the stick passes and when the tail end passes to get a Δt. Knowing the velocity of the stick we can then compute $L = v\Delta t \Rightarrow \Delta t = \frac{L}{v}$

So, if the stick is at rest in S' where 

then $\Delta x' = L_{rest}$

and in S we have data Δt, $\Delta x = 0$ $\Rightarrow$

$$\begin{pmatrix} c\Delta t \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} c\Delta t \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c\gamma\Delta t \\ -\beta\gamma c\Delta t \\ 0 \\ 0 \end{pmatrix}$$

Then $\underbrace{\Delta x'}_{= -L_{rest}} = -\underbrace{\beta}_{\beta = \frac{v}{c}}\gamma c\Delta t = -\frac{v}{c}\gamma c\frac{L}{v} = -\gamma L$

Why this (-) sign?

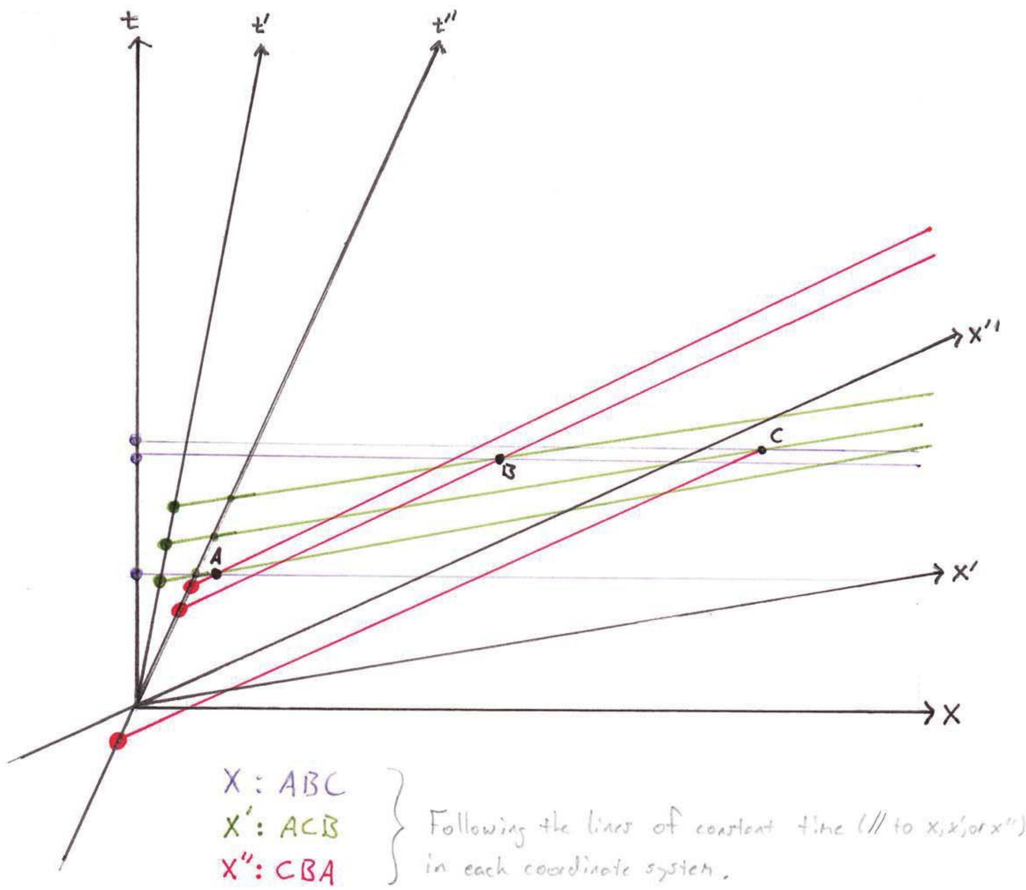
When I described measuring L_{rest} I moved from right to left, so $\Delta x' < 0$.

But length L is positive so $L_{rest} = -\Delta x'$.

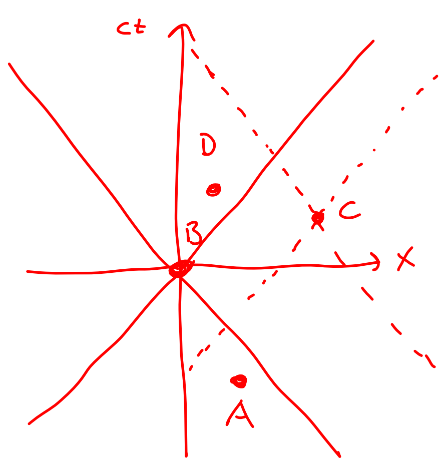
I moved from right to left because when we do the observation in S, we measure when the right end passes first and when the left end passes second, i.e. right to left!

So: $L = \frac{L_{rest}}{\gamma}$ and since $\gamma \geq 1$ this is length-contraction of moving objects.

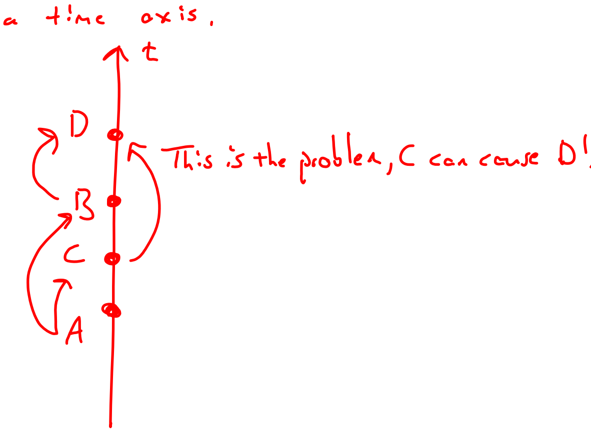
5. Three events **A,B,C** are seen by an observe O to occur in the order **ABC**. Another Observer O' sees the same three events occur in the order **CBA**. Is it possible that a third observer O'' could see the events in the order **ACB**? Support your conclusions by drawing a spacetime diagram.



6. On a *ct-x* spacetime diagram, draw four events **A**,**B**, **C** and **D** such that **A** can cause **B** and **C**, **B** can cause **D** but not **C**, and **C** cannot cause **D**. Is such a situation possible in Galilean Relativity?



In Galilean Relativity we essentially ignore spatial separation and use only a time axis.



7. Prove that in special relativity $(\Lambda^{0'}{}_0)^2 \geq 1$.

We know that $\underbrace{\Lambda^T \eta \Lambda}$ $= \eta$. If I write $\Lambda = \begin{pmatrix} \Lambda^{0'}{}_0 & \Lambda^{0'}{}_1 & \Lambda^{0'}{}_2 & \Lambda^{0'}{}_3 \\ \Lambda^{1'}{}_0 & \Lambda^{1'}{}_1 & \Lambda^{1'}{}_2 & \Lambda^{1'}{}_3 \\ \Lambda^{2'}{}_0 & \Lambda^{2'}{}_1 & \Lambda^{2'}{}_2 & \Lambda^{2'}{}_3 \\ \Lambda^{3'}{}_0 & \Lambda^{3'}{}_1 & \Lambda^{3'}{}_2 & \Lambda^{3'}{}_3 \end{pmatrix}$ $\eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

Then multiplying this gives a matrix, and if I look at the top left term it should be -1 since

This gives: $-(\Lambda^{0'}{}_0)^2 + (\Lambda^{1'}{}_0)^2 + (\Lambda^{2'}{}_0)^2 + (\Lambda^{3'}{}_0)^2 = -1$

or: $(\Lambda^{0'}{}_0)^2 = 1 + (\Lambda^{1'}{}_0)^2 + (\Lambda^{2'}{}_0)^2 + (\Lambda^{3'}{}_0)^2 \geq 1$

8. Consider objects N_{ij} and M^{ij} in 2D with components:

$N_{11} = a, N_{12} = b, N_{21} = c, N_{22} = d$ $\Rightarrow$ $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$M^{11} = e, M^{12} = f, M^{21} = g, M^{22} = h$ $\Rightarrow$ $M = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

Evaluate the following using index notation:

a) $N_{ij}M^{ki}$

b) $N_{ij}M^{kj}$

c) $N_{ij}M^{ji}$

d) $N_{ij}M^{ij}$

For each of the above, rewrite and evaluate using matrix operations when possible.

a) $N_{11}M^{11} + N_{21}M^{12} = ae + cf$ $\quad N_{11}M^{21} + N_{21}M^{22} = ag + ch$

$N_{12}M^{11} + N_{22}M^{12} = eb + df$ $\quad N_{12}M^{21} + N_{22}M^{22} = bg + dh$

or $N^T M^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} e & g \\ f & h \end{pmatrix} = \begin{pmatrix} ae + cf & ag + ch \\ be + df & bg + dh \end{pmatrix}$ labelled by $_j{}^k$ $\equiv$ or $MN = \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{pmatrix}$ labelled by $^k{}_j$

b) $N_{11}M^{11} + N_{12}M^{12} = ae + bf$ $\quad N_{11}M^{21} + N_{12}M^{22} = ag + bh$

$N_{21}M^{11} + N_{22}M^{12} = ce + df$ $\quad N_{21}M^{21} + N_{22}M^{22} = cg + dh$

or $NM^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & g \\ f & h \end{pmatrix} = \begin{pmatrix} ae + bf & ag + bh \\ ce + df & cg + dh \end{pmatrix}$

c) $N_{11}M^{11} + N_{12}M^{21} + N_{21}M^{12} + N_{22}M^{22} = ae + bg + cf + dh$

or $Tr(NM) = Tr\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) = Tr\begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} = ae + bg + cf + dh$

d) $N_{11}M^{11} + N_{21}M^{21} + N_{12}M^{12} + N_{22}M^{22} = ae + bf + cg + dh$

or $Tr(NM^T) = Tr\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & g \\ f & h \end{pmatrix}\right) = Tr\begin{pmatrix} ae + bf & ag + bh \\ ce + df & cg + dh \end{pmatrix} = ae + bf + cg + dh$