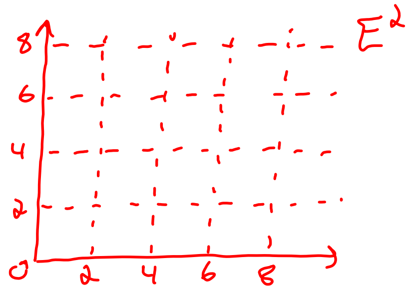
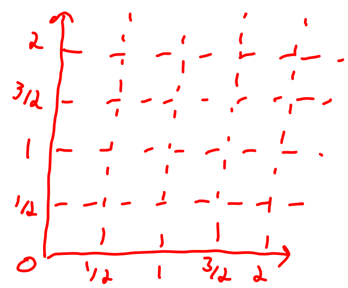
# General Relativity HW4 Problems

1. The notion of the tangent space and cotangent space at a point are connected through the metric on the space. This pair is one example of dual spaces, but there are many others. Here is another example of a dual space: Consider the lattice of points in $\mathbb{R}^2$ which are at even integer coordinate positions, i.e. $(x, y) = (2n_x, 2n_y)$ where $n_x, n_y$ are integers. We will call this the even lattice $E^2$. Now consider a dual lattice $D^2$ which is composed of points such that the Euclidean inner product of any lattice vector in $D^2$ with any lattice vector in $E^2$ always gives an integer. Identify the full set of points in this dual lattice. Note any differences between this dual pair and the tangent and cotangent spaces we have encountered.




Then:

$$V_{E^2} \cdot V_{D^2} = (2n_x, 2n_y)\begin{pmatrix} \frac{n_x}{2} \\ \frac{n_y}{2} \end{pmatrix}$$

$$= n_x + n_y \in \mathbb{Z}$$

$$V_{E^2} = (\underbrace{2n_x, 2n_y}) \qquad V_{D_2} = (\underbrace{\tfrac{n_x}{2}, \tfrac{n_y}{2}})$$

$n_x, n_y \in \mathbb{Z}$

integers

2. For a 3D space in a particular set of coordinates the metric takes the form $g_{\mu\nu} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$

   Note that we do not use $\eta$ for the metric unless we are in Minkowski space. More generally we will denote the metric by $g$.

   a) For a vector in this space with components (1,1,1) determine the components of the corresponding dual vector.

   b) For a dual vector with components (1,1,1) determine the components of the corresponding vector.

   c) Determine the "dot product" between the vector given in (a) with the dual vector given in (b).

   d) Determine the "dot product" between the vector given in (a) and its corresponding dual vector.

   e) Determine the "dot product" between the dual vector given in (b) and its corresponding vector.

a) $V^\mu = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow V_\mu = g_{\mu\nu} V^\nu = \begin{pmatrix} A & & \\ & B & \\ & & C \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

b) $W_\mu = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow W^\mu = g^{\mu\nu} W_\mu = \begin{pmatrix} A^{-1} & & \\ & B^{-1} & \\ & & C^{-1} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} A^{-1} \\ B^{-1} \\ C^{-1} \end{pmatrix}$

c) $V^\mu W_\mu = (1\ 1\ 1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3$ alternatively $V_\mu W^\mu = (A\ B\ C)\begin{pmatrix} A^{-1} \\ B^{-1} \\ C^{-1} \end{pmatrix} = 3$

d) $V_\mu V^\mu = (A\ B\ C)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = A + B + C$ alternatively $V^\mu V_\mu = (1\ 1\ 1)\begin{pmatrix} A \\ B \\ C \end{pmatrix} = A + B + C$

e) $W_\mu W^\mu = (1\ 1\ 1)\begin{pmatrix} A^{-1} \\ B^{-1} \\ C^{-1} \end{pmatrix} = A^{-1} + B^{-1} + C^{-1}$ alternatively $W^\mu W_\mu = (A^{-1} B^{-1} C^{-1})\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = A^{-1} + B^{-1} + C^{-1}$

3. Imagine we have a tensor $X^{\mu\nu}$ and a vector $V^\mu$ with components $X^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}$,

$V^\mu = (-1, \quad 2, \quad 0, \quad -2)$. In this problem you may assume the Minkowski metric diag(-1,1,1,1).

Find the components of:

a) $X^\mu{}_\nu$

b) $X_\mu{}^\nu$

c) $X^{(\mu\nu)}$

d) $X_{[\mu\nu]}$

e) $X^\lambda{}_\lambda$

f) $V^\mu V_\mu$

In this problem the relevant metric is that of Minkowski space in rectangular coordinates, i.e. $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$. Recall $\eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ and $\eta_{\mu\nu} = \eta_{\nu\mu}$.

Note: $(\mu\nu)$ means construct $\frac{1}{2}\mu\nu + \frac{1}{2}\nu\mu$ while $[\mu\nu]$ means construct $\frac{1}{2}\mu\nu - \frac{1}{2}\nu\mu$.

a) $X^\mu{}_\nu = \eta_{\nu\alpha} X^{\mu\alpha} = X^{\mu\alpha}\eta_{\alpha\nu} \Rightarrow \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix}$

b) $X_\mu{}^\nu = \eta_{\mu\kappa} X^{\kappa\nu} \Rightarrow \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}$ ⟵ Note these are different!

c) $X^{(\mu\nu)} = \frac{1}{2}(X^{\mu\nu} + X^{\nu\mu}) \Rightarrow \frac{1}{2}\begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 & -1 & -1 & -2 \\ 0 & 0 & 1 & 1 \\ 1 & 3 & 0 & 1 \\ -1 & 2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 2 & -1/2 & 0 & -3/2 \\ -1/2 & 0 & 2 & 3/2 \\ 0 & 2 & 0 & 1/2 \\ -3/2 & 3/2 & 1/2 & -2 \end{pmatrix}$ Notice it is symmetric!

d) $X_{[\mu\nu]} = \frac{1}{2}(X_{\mu\nu} - X_{\nu\mu}) = \frac{1}{2}(\eta_{\mu\alpha}X^{\alpha\beta}\eta_{\beta\nu} - \eta_{\nu\alpha}X^{\alpha\beta}\eta_{\beta\mu})$

$\Rightarrow \frac{1}{2}\left[\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} - \left\{\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\right\}^T\right]$

$= \frac{1}{2}\left[\begin{pmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ -1 & 3 & 0 & 1 \\ 1 & 2 & 0 & -2 \end{pmatrix}\right] = \begin{pmatrix} 0 & -1/2 & -1 & -1/2 \\ 1/2 & 0 & 1 & 1/2 \\ 1 & -1 & 0 & -1/2 \\ 1/2 & -1/2 & 1/2 & 0 \end{pmatrix}$ Notice it is antisymmetric!

e) $X^\lambda{}_\lambda = \delta_{\lambda\alpha}\underbrace{X^\alpha{}_\lambda}_{\text{from part (a)}} \Rightarrow \mathrm{Tr}\begin{pmatrix} -2 & 0 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix} = -2-2 = -4$

f) $V^\mu V_\mu = V^\mu g_{\mu\nu} V^\nu \Rightarrow (-1, 2, 0, -2)\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 0 \\ -2 \end{pmatrix} = (-1, 2, 0, -2)\begin{pmatrix} 1 \\ 2 \\ 0 \\ -2 \end{pmatrix} = 7$

g) $V_\mu X^{\mu\nu} = g_{\mu\alpha}V^\alpha X^{\mu\nu} = V^\alpha g_{\alpha\mu} X^{\mu\nu} \Rightarrow (-1, 2, 0, -2)\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} = (4, -2, 5, 7)$

4. Starting from $\partial_{[\mu}F_{\nu\lambda]} = 0$ derive the corresponding Maxwell's equations in terms of 3-component vector quantities $\vec{E}$ and $\vec{B}$.

$$\partial_{[\mu}F_{\nu\lambda]} = \frac{1}{3!}\left\{\partial_\mu F_{\nu\lambda} + \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} - \partial_\mu F_{\lambda\nu} - \partial_\lambda F_{\nu\mu} - \partial_\nu F_{\mu\lambda}\right\} = 0$$

ignore since the expression $=0$ anyway so everything in { } must vanish.

We can choose:

| $\mu$ | $\nu$ | $\lambda$ |
|---|---|---|
| 0 | 1 | 2 |
| 0 | 1 | 3 |
| 0 | 2 | 3 |
| 1 | 2 | 3 |

All other index assignments either automatically vanish since they repeat an index, e.g. 011, or they simply reproduce one of these results with the terms in different order, e.g. 021.

Recall: $F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$ where $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$

so for example $F_{12} = B_z$, etc.

For 012: $\partial_0 F_{12} + \partial_2 F_{01} + \partial_1 F_{20} - \partial_0 F_{21} - \partial_2 F_{10} - \partial_1 F_{02} = 0$

$$\frac{\partial}{\partial t}(B_z) + \frac{\partial}{\partial y}(-E_x) + \frac{\partial}{\partial x}(E_y) - \frac{\partial}{\partial t}(-B_z) - \frac{\partial}{\partial y}(E_x) - \frac{\partial}{\partial x}(-E_y) = 0$$

$$2\left[\frac{\partial B_z}{\partial t} + \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)\right] = 0$$

Compare to $\underbrace{\left(\frac{\partial \vec{B}}{\partial t} + \vec{\nabla}\times\vec{E}\right)_z}_{\text{z-component of expression}} = 0$ where $\vec{\nabla}\times\vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{i} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{j} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{k}$

The 013 and 023 choices will simply give the y and x components of this equation respectively.

For 123: $\partial_1 F_{23} + \partial_3 F_{12} + \partial_2 F_{31} - \partial_1 F_{32} - \partial_3 F_{21} - \partial_2 F_{13} = 0$

$$\frac{\partial}{\partial x}(B_x) + \frac{\partial}{\partial z}(B_z) + \frac{\partial}{\partial y}(B_y) - \frac{\partial}{\partial x}(-B_x) - \frac{\partial}{\partial z}(-B_z) - \frac{\partial}{\partial y}(-B_y) = 0$$

$$2\left[\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right] = 0$$

$$2\,\vec{\nabla}\cdot\vec{B} = 0 \quad \checkmark$$