General Relativity HW4 Problems

1. Imagine a particle following a path through spacetime given by $x^\mu(\tau) = \left(\tau^2 + \tau, \tau^2, \frac{4}{3}\tau^{\frac{3}{2}}, -10\right)$.
   a) Compute the four-velocity of the particle as it passes through the point $x^\mu = (20,16,\frac{32}{3},-10)$.
   b) For the function $f(t,x,y,z) = -t^2 + x^2 + y^2 - yz$, calculate the rate of change of this function along the path from part (a), i.e. $\frac{df}{d\tau}$, at the point $x^\mu = (20,16,\frac{32}{3},-10)$.
   Hint: You will need to break up the derivative into two terms using ∂x^μ in various places so that can use your result for the four-velocity.
2. Consider the example in class where I found the vector which defined the area of a two-sphere in 3D. In this case, I want you to do the same but this time for a cube of side length 1 which is centered on the origin with edges along the coordinates. You will need an equation which defines the surface, then proceed as I did in class. The answer should be obvious once you get it.
3. The energy-momentum tensor of a perfect fluid in its rest frame is given by $T^{\mu\upsilon}$=diag(ρ,p,p,p). Find a matrix expression for the energy-momentum tensor seen by an observer moving with a speed v along the $e_{(1)}$+$e_{(2)}$ direction. Do this in two ways:
   a) Use Lorentz transformations to explicitly transform $T^{\mu\upsilon}$.
   b) Use the expression (valid in any frame) $T^{\mu\upsilon}=(\rho+p)U^\mu U^\nu+p\eta^{\mu\nu}$.
4. Most energy-momentum tensors naturally specify a preferred inertial frame for which the overall system is at rest. For the perfect fluid case, this is typically the frame for which the matrix realization of the tensor is diagonal. Consider the case of vacuum energy with an equation of state p=-ρ. Treating this as perfect fluid, what can you say about the preferred rest frame of the vacuum energy system?
5. Argue that naively adding a finite speed of propagation to Newtonian gravity will result in stable orbits becoming unstable. A simple scenario to consider is two equal mass objects in a circular orbit around a common point under their mutual gravitational interaction.
6. Suppose that we lived in a world with only two forces: gravity and the electromagnetic force. Also suppose that every bit of matter in this world was “extremal”, i.e. the electromagnetic charge of any particle is always equal to its mass (with a **universal** constant to get the dimensions correct). In this context, does it make sense to single out gravity as providing the “curvature of spacetime”, or could we instead use electromagnetism, or perhaps both?