General Relativity HW4 Problems

1. Imagine a particle following a path through spacetime given by $x^\mu(\tau) = \left(\tau^2 + \tau, \tau^2, \frac{4}{3}\tau^{\frac{3}{2}}, -10\right)$.

   a) Compute the four-velocity of the particle as it passes through the point $x^\mu = (20, 16, \frac{32}{3}, -10)$.

First note that the point under consideration corresponds to $\tau = 4$.

Then we want $U^\mu(\tau = 4) = \left.\frac{dx^\mu}{d\tau}\right|_{\tau=4}$

$$U^\mu(\tau) = (2\tau + 1, 2\tau, 2\tau^{1/2}, 0)$$

Thus:

$$U^\mu(\tau = 4) = (9, 8, 4, 0)$$

   b) For the function $f(t, x, y, z) = -t^2 + x^2 + y^2 - yz$, calculate the rate of change of this function along the path, i.e. $\frac{\partial f}{\partial \tau}$, at the point $x^\mu = (20, 16, \frac{32}{3}, -10)$.
   Hint: You will need to break up the directional derivative into two terms using ∂x^μ in various places so that can use your result for the four-velocity.

To evaluate $\frac{\partial f}{\partial \tau}$ consider $\frac{\partial f}{\partial \tau} = \frac{\partial f}{\partial x^\mu}\frac{\partial x^\mu}{\partial \tau} = \frac{\partial f}{\partial x^\mu}\underbrace{U^\mu}_{\text{from part (a)}}$

First:

$$\frac{\partial f}{\partial x^\mu} = \left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$= (-2t, 2x, 2y - z, -y)$$

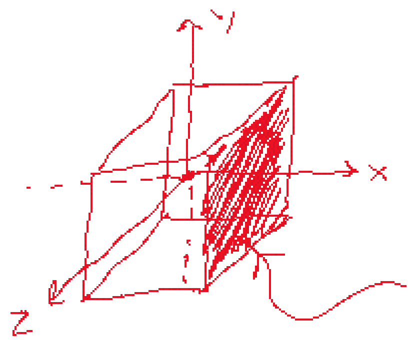
Then:

$$\frac{\partial f}{\partial x^\mu}U^\mu = (-2t, 2x, 2y - z, -y)\begin{pmatrix} 2\tau+1 \\ 2\tau \\ 2\tau^{1/2} \\ 0 \end{pmatrix} = -2t(2\tau+1) + 4x\tau + 4y\tau^{1/2} - 2z\tau^{1/2}$$

Then at $\tau = 4$ or $t = 20$, $x = 16$, $y = \frac{32}{3}$, $z = -10$ we have

$$\left.\frac{\partial f}{\partial \tau}\right|_{\tau=4} = -40(9) + 256 + 85.33 + 40 = 21.33$$

2. Consider the example in class where I found the vector which defined the area of a two-sphere in 3D. In this case, I want you to do the same but this time for a cube of side length 1 which is centered on the origin with edges along the coordinates. You will need an equation which defines the surface, then proceed as I did in class. The answer should be obvious once you get it.



This surface is described by $x = \frac{1}{2}$

which gives $f(x,y,z) = x - \frac{1}{2} = 0$.

Then the normal (dual) vector is:

$\vec{N} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = \hat{i}$ as expected.

The other surface normals play out similarly.

3. The energy-momentum tensor of a perfect fluid in its rest frame is given by $T^{\mu\nu}=\mathrm{diag}(\rho,p,p,p)$. Find a matrix expression for the energy-momentum tensor seen by an observer moving with a speed v along the $e_{(1)}+e_{(2)}$ direction. Do this in two ways:

a) Use Lorentz transformations to explicitly transform $T^{\mu\nu}$.

We need the matrix form of a boost along $e_{(1)}+e_{(2)}$. To get this, we can first rotate $\overset{y}{\llcorner}{}_{x}$ by 45° to x', then boost along x' (which we know the form for), then rotate back to $\overset{y}{\llcorner}{}_{x}$ to see what the result looks like in the original spatial coordinates.

Before we do that, let's consider how $T^{\mu\nu}$ transforms in general.

$$T^{\mu\nu} \to T^{\mu'\nu'} = \Lambda^{\mu'}{}_{\mu}\Lambda^{\nu'}{}_{\nu}T^{\mu\nu} = \Lambda^{\mu'}{}_{\mu}T^{\mu\nu}\Lambda^{\nu'}{}_{\nu} = \Lambda T \Lambda^{T} \text{ w/ } \Lambda = \Lambda^{\mu'}{}_{\mu}$$

Doing our transformation in 3 steps we find:

$$T^{\mu\nu} \to T^{\mu'\nu'} = \Lambda_{R_{xy}(-45°)}\Lambda_{B_{xt}}\underbrace{\Lambda_{R_{xy}(45°)}\,T\,\Lambda^{T}_{R_{xy}(45°)}}\Lambda^{T}_{B_{xt}}\Lambda^{T}_{R_{xy}(-45°)}$$

Since $T^{\mu\nu}_{rest} = \begin{pmatrix}\rho & & & \\ & p & & \\ & & p & \\ & & & p\end{pmatrix}$ which is isotropic in x,y,z, so we know that $R_{xy}(45°)$ will not change anything! You don't have to realize this. Just multiply it out!

Then using $\Lambda_{B_{xt}} = \begin{pmatrix}\gamma & -\gamma v & 0 & 0\\ -\gamma v & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$ and $\Lambda_{R_{xy}(-45°)} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$

We have:

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\gamma & -\gamma v & 0 & 0\\ -\gamma v & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\rho & 0 & 0 & 0\\ 0 & p & 0 & 0\\ 0 & 0 & p & 0\\ 0 & 0 & 0 & p\end{pmatrix}\begin{pmatrix}\gamma & -\gamma v & 0 & 0\\ -\gamma v & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}\gamma & -\gamma v & 0 & 0\\ -\frac{\gamma v}{\sqrt{2}} & \frac{\gamma}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ -\frac{\gamma v}{\sqrt{2}} & \frac{\gamma}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\rho & & & \\ & p & & \\ & & p & \\ & & & p\end{pmatrix}\begin{pmatrix}\gamma & -\frac{\gamma v}{\sqrt{2}} & -\frac{\gamma v}{\sqrt{2}} & 0\\ -\gamma v & \frac{\gamma}{\sqrt{2}} & \frac{\gamma}{\sqrt{2}} & 0\\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$T^{\mu'\nu'} = \begin{pmatrix}\gamma & -\gamma v & 0 & 0\\ -\frac{\gamma v}{\sqrt{2}} & \frac{\gamma}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ -\frac{\gamma v}{\sqrt{2}} & \frac{\gamma}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\rho\gamma & -\rho\frac{\gamma v}{\sqrt{2}} & -\rho\frac{\gamma v}{\sqrt{2}} & 0\\ -p\gamma v & p\frac{\gamma}{\sqrt{2}} & p\frac{\gamma}{\sqrt{2}} & 0\\ 0 & -p\frac{1}{\sqrt{2}} & p\frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 0 & p\end{pmatrix} = \begin{pmatrix}\rho\gamma^2+pv^2\gamma^2 & -\rho\frac{v}{\sqrt{2}}\gamma^2-p\frac{v}{\sqrt{2}}\gamma^2 & -\rho\frac{v}{\sqrt{2}}\gamma^2-p\frac{v}{\sqrt{2}}\gamma^2 & 0\\ -\rho\frac{v}{\sqrt{2}}\gamma^2-p\frac{v}{\sqrt{2}}\gamma^2 & \rho\frac{v^2}{2}\gamma^2+p\frac{\gamma^2}{2}+\frac{p}{2} & \rho\frac{v^2}{2}\gamma^2+p\frac{\gamma^2}{2}-\frac{p}{2} & 0\\ -\rho\frac{v}{\sqrt{2}}\gamma^2-p\frac{v}{\sqrt{2}}\gamma^2 & \rho\frac{v^2}{2}\gamma^2+p\frac{\gamma^2}{2}-\frac{p}{2} & \rho\frac{v^2}{2}\gamma^2+p\frac{\gamma^2}{2}+\frac{p}{2} & 0\\ 0 & 0 & 0 & p\end{pmatrix}$$

For comparison to the next part, let's rewrite a few terms using $\gamma^2 = \frac{1}{1-v^2}$ or $v^2 = 1-\frac{1}{\gamma^2}$:

$$T^{0'0'} = \rho\gamma^2 + pv^2\gamma^2 = \rho\gamma^2 + p\gamma^2 - p$$

$$T^{1'1'} = T^{2'2'} = \rho\frac{v^2}{2}\gamma^2 + p\frac{\gamma^2}{2} + \frac{p}{2} = \rho\frac{v^2}{2}\gamma^2 + \frac{p}{2}\gamma^2 - \frac{p}{2} + p = \rho\frac{v^2}{2}\gamma^2 + \frac{p}{2}\gamma^2\left(1-\frac{1}{\gamma^2}\right) + p = \rho\frac{v^2}{2}\gamma^2 + \frac{p}{2}\gamma^2 v^2 + p$$

$$T^{2'1'} = T^{1'2'} = \rho\frac{v^2}{2}\gamma^2 + p\frac{\gamma^2}{2} - \frac{p}{2} = \rho\frac{v^2}{2}\gamma^2 + \frac{p}{2}\gamma^2\left(1-\frac{1}{\gamma^2}\right) = \rho\frac{v^2}{2}\gamma^2 + p\frac{v^2\gamma^2}{2}$$

b) Use the expression (valid in any frame) $T^{\mu\nu}=(\rho+p)U^{\mu}U^{\nu}+p\eta^{\mu\nu}$.

Using that an observer moving along $e_{(1)}+e_{(2)}$ w/ speed v sees the fluid moving along $-(e_{(1)}+e_{(2)})$ w/ v:

$$U^{\mu} = \begin{pmatrix}\gamma\\ -\frac{v}{\sqrt{2}}\gamma\\ -\frac{v}{\sqrt{2}}\gamma\\ 0\end{pmatrix}$$

4-velocity of fluid

Then: $$T^{\mu'\nu'} = (\rho+p)\begin{pmatrix}\gamma^2 & -\frac{v}{\sqrt{2}}\gamma^2 & -\frac{v}{\sqrt{2}}\gamma^2 & 0\\ -\frac{v}{\sqrt{2}}\gamma^2 & \frac{v^2}{2}\gamma^2 & \frac{v^2}{2}\gamma^2 & 0\\ -\frac{v}{\sqrt{2}}\gamma^2 & \frac{v^2}{2}\gamma^2 & \frac{v^2}{2}\gamma^2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} + p\begin{pmatrix}-1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{pmatrix} = \begin{pmatrix}\rho\gamma^2+p\gamma^2-p & -\rho\frac{v}{\sqrt{2}}\gamma^2-p\frac{v}{\sqrt{2}}\gamma^2 & -\rho\frac{v}{\sqrt{2}}\gamma^2-p\frac{v}{\sqrt{2}}\gamma^2 & 0\\ -\rho\frac{v}{\sqrt{2}}\gamma^2-p\frac{v}{\sqrt{2}}\gamma^2 & \rho\frac{v^2}{2}\gamma^2+p\frac{v^2}{2}\gamma^2+p & \rho\frac{v^2}{2}\gamma^2+p\frac{v^2}{2}\gamma^2 & 0\\ -\rho\frac{v}{\sqrt{2}}\gamma^2-p\frac{v}{\sqrt{2}}\gamma^2 & \rho\frac{v^2}{2}\gamma^2+p\frac{v^2}{2}\gamma^2 & \rho\frac{v^2}{2}\gamma^2+p\frac{v^2}{2}\gamma^2+p & 0\\ 0 & 0 & 0 & p\end{pmatrix}$$

4.   Most energy-momentum tensors naturally specify a preferred inertial frame for which the overall system is at rest. For the perfect fluid case, this is typically the frame for which the matrix realization of the tensor is diagonal. Consider the case of vacuum energy with an equation of state p=-ρ.   Treating this as perfect fluid, what can you say about the preferred rest frame of the vacuum energy system?

For vacuum we have $T^{\mu\nu}_{rest} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} = \begin{pmatrix} -p & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} = p\,\eta^{\mu\nu}$

But we know that SR is based on transformations between (inertial) frames which leave $\eta_{\mu\nu}$ (hence $\eta^{\mu\nu}$) invariant, so we find that $T^{\mu\nu}_{vac}$ is also invariant.

This means that unlike a normal perfect fluid where it means something to be at rest w.r.t. the fluid (and hence $T^{\mu\nu}$ transforms to another form when boosted), it doesn't make any sense to be at rest w.r.t. the vacuum. All observers are equal w.r.t. to the vacuum, so they should (and do!) see the same form for $T^{\mu\nu}_{vac}$.

5. Argue that naively adding a finite speed of propagation to Newtonian gravity will result in stable orbits becoming unstable. A simple scenario to consider is two equal mass objects in a circular orbit around a common point under their mutual gravitational interaction.

6. Suppose that we lived in a world with only two forces: gravity and the electromagnetic force. Also suppose that every bit of matter in this world was “extremal”, i.e. the electromagnetic charge of any particle is always equal to its mass (with a **universal** constant to get the dimensions correct). In this context, does it make sense to single out gravity as providing the “curvature of spacetime”, or could we instead use electromagnetism, or perhaps both?

The curvature of spacetime effects the trajectory of all particles that move on it whether they or massive or not. However we know that the electromagnetic interaction does not effect electrically neutral particles, so it does not have the same "universality" that gravitation does, and hence could not be represented by something as universal as the curvature of spacetime.

Sidenote: The electromagnetic interaction can be given an interpretation in terms of curvature, however the curvature belongs to a U(1) fiber bundle over spacetime.