General Relativity HW 5 Quiz

Name ___KEY___

You know the drill!

1. (*10pts*) Prolate spheroidal coordinates are related to the usual Cartesian coordinates $\{x, y, x\}$ of Euclidean three-space by

$$x = \sinh\chi \sin\theta \cos\phi$$
$$y = \sinh\chi \sin\theta \sin\phi$$
$$z = \cosh\chi \cos\theta$$

What does the invariant interval ds^2 look like in prolate spheroidal coordinates when $\theta = \frac{\pi}{2}$?

When $\theta = \frac{\pi}{2}$ we have $x = \sinh\chi\cos\phi$, $y = \sinh\chi\sin\phi$, $z = 0$

There are 2 ways to find ds^2:

a) $dx = \frac{\partial x}{\partial \chi} d\chi + \frac{\partial x}{\partial \phi} d\phi = \cosh\chi\cos\phi\, d\chi - \sinh\chi\sin\phi\, d\phi$

$dy = \frac{\partial y}{\partial \chi} d\chi + \frac{\partial y}{\partial \phi} d\phi = \cosh\chi\sin\phi\, d\chi + \sinh\chi\cos\phi\, d\phi$

Then $ds^2 = dx^2 + dy^2 = (\cosh\chi\cos\phi\, d\chi - \sinh\chi\sin\phi\, d\phi)^2 + (\cosh\chi\sin\phi\, d\chi + \sinh\chi\cos\phi\, d\phi)^2$

$$ds^2 = \cosh^2\chi\cos^2\phi\, d\chi^2 + \sinh^2\chi\sin^2\phi\, d\phi^2 \cancel{- 2\cosh\chi\cos\phi\sinh\chi\sin\phi\, d\chi d\phi}$$
$$+ \cosh^2\chi\sin^2\phi\, d\chi^2 + \sinh^2\chi\cos^2\phi\, d\phi^2 \cancel{+ 2\cosh\chi\sin\phi\sinh\chi\cos\phi\, d\chi d\phi}$$
$$= \cosh^2\chi\, d\chi^2 + \sinh^2\chi\, d\phi^2$$

b) $\underbrace{g_{\mu\nu} \to g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}}$ w/ $\frac{\partial x^\mu}{\partial x^{\mu'}} = \begin{pmatrix} \frac{\partial x}{\partial \chi} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \chi} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \cosh\chi\cos\phi & -\sinh\chi\sin\phi \\ \cosh\chi\sin\phi & \sinh\chi\sin\phi \end{pmatrix}$

$x^\mu = (\chi, \phi) \to x^{\mu'} = (x, y)$

Then $g_{\mu'\nu'} = \begin{pmatrix} \cosh\chi\cos\phi & -\sinh\chi\sin\phi \\ \cosh\chi\sin\phi & \sinh\chi\sin\phi \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\chi\cos\phi & -\sinh\chi\sin\phi \\ \cosh\chi\sin\phi & \sinh\chi\cos\phi \end{pmatrix}$

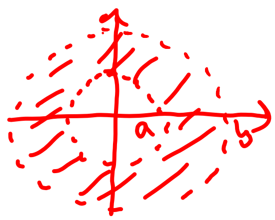
$$= \begin{pmatrix} \cosh^2\chi\cos^2\phi + \cosh^2\chi\sin^2\phi & 0 \\ 0 & \sinh^2\chi\sin^2\phi + \sinh^2\chi\cos^2\phi \end{pmatrix} = \begin{pmatrix} \cosh^2\chi & 0 \\ 0 & \sinh^2\chi \end{pmatrix}$$

Finally, $ds^2 = (d\chi\ d\phi)\, g \begin{pmatrix} d\chi \\ d\phi \end{pmatrix} = \cosh^2\chi\, d\chi^2 + \sinh^2\chi\, d\phi^2$

2. (*10pts*) Consider the open annulus which is the set of points in $\mathbb{R}^2$ such that $a < r < b$, when $\mathbb{R}^2$ is described in terms of polar coordinates (r, θ). Show that this space is a manifold that can be covered by a single chart. In your answer make sure you provide the explicit chart map.

The open annulus is $r \in (a, b)$, $\theta \in (0, 2\pi]$ in $\mathbb{R}^2$

or



the region outside of a circle of radius a, but inside a circle of radius b.

For a chart we only need a map from points in the annulus into $\mathbb{R}^2$, but the annulus itself is defined in $\mathbb{R}^2$ so the identity map will do!