General Relativity HW6 Problems

1. As a warm-up to the GR case, you will determine the transformation rule for the electromagnetic gauge field as a result of demanding covariance of the Dirac equation. Don't worry if a lot of these words don't make sense initially. In fact you will not really need to understand most of them to do the what I am asking.

   a. Consider the equation $\gamma^\mu \partial_\mu \psi + \frac{mc}{\hbar}\psi = 0$ where $\psi(x^\mu)$ is a spinor field and γ^μ are the constant Dirac matrices. The rest of the terms in this equation should be familiar. This equation is "covariant" under a transformation of the form $\psi \to \psi' = e^{iq\varphi}\psi$ where q and φ are constants. What covariant means is that the **entire** left hand side of the equation transforms as $\gamma^\mu \partial_\mu \psi + \frac{mc}{\hbar}\psi \to e^{iq\varphi}(\gamma^\mu \partial_\mu \psi + \frac{mc}{\hbar}\psi)$ which you can easily verify. Now what we want to do is make this equation covariant even when we allow φ to depend on position, i.e. $\varphi(x^\mu)$. You can check that in its current form this equation is not covariant under this "local" transformation since $\partial_\mu \psi \to \partial_\mu(e^{iq\varphi(x^\mu)}\psi)$ and the derivative will now act on both $\varphi(x^\mu)$ and $\psi(x^\mu)$. To fix this, we will make use of a new derivative of the form $D_\mu \equiv \partial_\mu + iqA_\mu$ where $A_\mu(x^\mu)$ will end up being the electromagnetic 4-vector potential (or gauge field).

   Your job is to figure out how the gauge field itself must transform in order for the equation with the new derivative to be covariant. That is, what does $A'_\mu(x^\mu)$ look like in terms of $A_\mu(x^\mu)$ and other quantities such that $\gamma^\mu D_\mu \psi + \frac{mc}{\hbar}\psi \to e^{iq\varphi(x^\mu)}(\gamma^\mu D_\mu \psi + \frac{mc}{\hbar}\psi)$.

a) $\gamma^\mu \partial_\mu \psi + \frac{mc}{\hbar}\psi = 0$

$\Downarrow$

we already know that this term is gonna do what we want, so let's leave it out

$$\gamma^\mu D_\mu \psi + \frac{mc}{\hbar}\psi = \gamma^\mu(\partial_\mu + iqA_\mu)\psi + \frac{mc}{\hbar}\psi = 0$$

Transforming we have:

$\gamma^\mu(\partial_\mu + iqA'_\mu)\psi'$ where $\psi' = e^{iq\phi(x^\mu)}\psi$

$\gamma^\mu(\partial_\mu + iqA'_\mu)e^{iq\phi(x^\mu)}\psi$ we want this to be: $\boxed{e^{iq\phi(x^\mu)}\gamma^\mu(\partial_\mu + iqA_\mu)\psi}$

$$\gamma^\mu \underbrace{\partial_\mu(e^{iq\phi(x^\mu)}\psi)}_{\text{product rule}} + \gamma^\mu iqA'_\mu e^{iq\phi(x^\mu)}\psi$$

$$\gamma^\mu e^{iq\phi(x^\mu)} iq(\partial_\mu \phi(x^\mu))\psi + \cancel{\gamma^\mu e^{iq\phi(x^\mu)}\partial_\mu\psi} + \gamma^\mu iqA'_\mu e^{iq\phi(x^\mu)}\psi$$

$$= \cancel{e^{iq\phi(x^\mu)}\gamma^\mu\partial_\mu\psi} + e^{iq\phi(x^\mu)}\gamma^\mu iqA_\mu\psi$$

Rearranging: $\gamma^\mu iqA'_\mu e^{iq\phi(x^\mu)}\psi = e^{iq\phi(x^\mu)}\gamma^\mu iqA_\mu\psi - \gamma^\mu e^{iq\phi(x^\mu)} iq(\partial_\mu\phi(x^\mu))\psi$

$\Rightarrow A'_\mu = A_\mu - \partial_\mu \phi(x^\mu)$ which is the usual result!

b. Now, turning to GR, derive the transformation rule for the connection $\Gamma^{\nu}_{\mu\lambda}$ from insisting that $\partial_\mu V^\nu$ be a tensorial (or "covariant" derivative).

b. For $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\lambda} V^\lambda$ we want $\nabla_\mu V^\nu \to \nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu$

So consider:

$$\nabla_{\mu'} V^{\nu'} = \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda'}$$

We know $\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu$, $V^{\nu'} = \frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu$ (same for λ') so:

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \left(\frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu \right) + \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda$$

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu} V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^{\lambda'}}{\partial x^\lambda} \Gamma^{\nu'}_{\mu'\lambda'} V^\lambda$$

We want this to be equal to $\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} (\partial_\mu V^\nu + \Gamma^{\nu}_{\mu\lambda} V^\lambda)$

Setting these equal:

$$\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu} V^\nu + \cancel{\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu} + \frac{\partial x^{\lambda'}}{\partial x^\lambda} \Gamma^{\nu'}_{\mu'\lambda'} V^\lambda = \cancel{\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu} + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^{\nu}_{\mu\lambda} V^\lambda$$

Solving for $\Gamma^{\nu'}_{\mu'\nu'}$:

$$\frac{\partial x^{\lambda'}}{\partial x^\lambda} \Gamma^{\nu'}_{\mu'\lambda'} V^\lambda = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^{\nu}_{\mu\lambda} V^\lambda - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu} V^\nu$$

rename this λ

$$\Gamma^{\nu'}_{\mu'\lambda'} V^\lambda = \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^{\nu}_{\mu\lambda} V^\lambda - \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda} V^\lambda$$

and multiply both sides w/ $\frac{\partial x^\lambda}{\partial x^{\lambda'}}$

Ignoring the common V^λ we are left w/

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^{\nu}_{\mu\lambda} - \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}$$

2. Imagine you have a diagonal metric $g_{\mu\nu}$. Show that the Christoffel connection components are given by:

$$\Gamma^{\lambda}_{\mu\nu} = 0$$
$$\Gamma^{\lambda}_{\mu\mu} = -\frac{1}{2}(g_{\lambda\lambda})^{-1}\partial_{\lambda} g_{\mu\mu}$$
$$\Gamma^{\lambda}_{\mu\lambda} = \partial_{\mu}(ln\sqrt{|g_{\lambda\lambda}|})$$
$$\Gamma^{\lambda}_{\lambda\lambda} = \partial_{\lambda}(ln\sqrt{|g_{\lambda\lambda}|})$$

where in these expressions $\mu \neq \nu \neq \lambda$ and repeated indices are **not** summed over. **Hint:** Use the expression we obtained for the Christoffel components in terms of derivatives of the metric and assign index values as needed.

Note a couple of useful things about diagonal metrics:

$g_{\mu\nu} = 0$ if $\mu \neq \nu$, $g^{\mu\nu} = 0$ if $\mu \neq \nu$, $g_{\mu\nu} = \begin{pmatrix} g_{11} & & \\ & g_{22} & \\ & & \ddots \end{pmatrix} \Rightarrow g^{\mu\nu} = \begin{pmatrix} \frac{1}{g_{11}} & & \\ & \frac{1}{g_{22}} & \\ & & \ddots \end{pmatrix}$

$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$ general expression for $\Gamma^{\lambda}_{\mu\nu}$

a) $\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} \underbrace{g^{\lambda\rho}}_{=0 \text{ unless } \rho=\lambda}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) = \frac{1}{2} g^{\lambda\lambda}(\partial_\mu \underbrace{g_{\nu\lambda}}_{=0\ \nu\neq\lambda} + \partial_\nu \underbrace{g_{\lambda\mu}}_{=0\ \lambda\neq\mu} - \partial_\lambda \underbrace{g_{\mu\nu}}_{=0\ \mu\neq\nu})$

$= 0$

b) $\Gamma^{\lambda}_{\mu\mu} = \frac{1}{2} \underbrace{g^{\lambda\rho}}_{=0 \text{ unless } \rho=\lambda}(\partial_\mu g_{\mu\rho} + \partial_\mu g_{\rho\mu} - \partial_\rho g_{\mu\mu}) = \frac{1}{2} g^{\lambda\lambda}(\partial_\mu \underbrace{g_{\mu\lambda}}_{=0} + \partial_\mu \underbrace{g_{\lambda\mu}}_{=0} - \partial_\lambda g_{\mu\mu})$

$\mu \neq \lambda$

$= -\frac{1}{2} g^{\lambda\lambda} \partial_\lambda g_{\mu\mu} = -\frac{1}{2}\frac{1}{g_{\lambda\lambda}}\partial_\lambda g_{\mu\mu}$

c) $\Gamma^{\lambda}_{\mu\lambda} = \frac{1}{2} \underbrace{g^{\lambda\rho}}_{=0 \text{ unless } \rho=\lambda}(\partial_\mu g_{\lambda\rho} + \partial_\lambda g_{\rho\mu} - \partial_\rho g_{\mu\lambda}) = \frac{1}{2} g^{\lambda\lambda}(\partial_\mu g_{\lambda\lambda} + \partial_\lambda \underbrace{g_{\lambda\mu}}_{=0} - \partial_\lambda \underbrace{g_{\mu\lambda}}_{=0})$

$\mu \neq \lambda$

$= \frac{1}{2}\frac{1}{g_{\lambda\lambda}}\partial_\mu g_{\lambda\lambda} = \frac{1}{2}\partial_\mu \ln g_{\lambda\lambda} = \partial_\mu \ln(|g_{\lambda\lambda}|)^{\frac{1}{2}}$

d) $\Gamma^{\lambda}_{\lambda\lambda} = \frac{1}{2} \underbrace{g^{\lambda\rho}}_{=0 \text{ unless } \lambda=\rho}(\partial_\lambda g_{\lambda\rho} + \partial_\lambda g_{\rho\lambda} - \partial_\rho g_{\lambda\lambda}) = \frac{1}{2} g^{\lambda\lambda}(\partial_\lambda g_{\lambda\lambda} + \cancel{\partial_\lambda g_{\lambda\lambda}} - \cancel{\partial_\lambda g_{\lambda\lambda}})$

$= \frac{1}{2}\frac{1}{g_{\lambda\lambda}}\partial_\lambda g_{\lambda\lambda} = \frac{1}{2}\partial_\mu \ln g_{\lambda\lambda} = \partial_\mu \ln(|g_{\lambda\lambda}|)^{\frac{1}{2}}$

3. You may be familiar with the gradient $\vec{\nabla}\varphi$ and divergence $\vec{\nabla}\cdot\vec{V}$ in spherical polar coordinates from E&M. As a reminder one can relate Cartesian to spherical polar coordinates via

$$x = r\,sin\theta\,cos\phi$$
$$y = r\,sin\theta\,sin\phi$$
$$z = r\,cos\theta$$

The metric in Cartesian coordinates, i.e. $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then takes the form $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 sin^2\theta \end{pmatrix}$. The operator $\vec{\nabla}$ that you saw in your E&M class was secretly just the covariant derivative that we have introduced! Given this, and the form of the metric in spherical polar coordinates, derive expressions for the gradient of a scalar field $\vec{\nabla}\varphi$ and divergence of a vector field $\vec{\nabla}\cdot\vec{V}$ in spherical polar coordinates. Compare your answers to what you see in E&M textbooks and note any discrepancies. **Hint:** You can use Mathematica to get the Christoffel components if you like (download the GREAT package and have a look) or you can use the results of the previous problem to get them by hand.

We can compute the Christoffel components using the previous problem. To do so let's first get some useful quantities:

$g_{rr} = 1 \quad g_{\theta\theta} = r^2 \quad g_{\phi\phi} = r^2 \sin^2\theta$

$\partial_r g_{\theta\theta} = 2r \quad \partial_r g_{\phi\phi} = 2r\sin^2\theta$

$\partial_\theta g_{\phi\phi} = 2r^2 \sin\theta\cos\theta$

Then using $i,j,k \in \{r,\theta,\phi\}$ we get:

$\Gamma^i_{jk} = 0 \Rightarrow \Gamma^r_{\theta\phi} = \Gamma^\theta_{r\phi} = \Gamma^\phi_{\theta r} = 0$ or any switch of lower indices

$\Gamma^i_{jj} = -\frac{1}{2}\frac{1}{g_{ii}}\partial_i g_{jj} \Rightarrow \Gamma^r_{\theta\theta} = -r, \quad \Gamma^r_{\phi\phi} = -r\sin^2\theta, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$

$\Gamma^i_{ji} = \partial_j\left(\ln\sqrt{|g_{ii}|}\right) \Rightarrow \Gamma^\theta_{r\theta} = \frac{1}{r}, \quad \Gamma^\phi_{r\phi} = \frac{1}{r}, \quad \Gamma^\phi_{\theta\phi} = \cot\theta$

$\Gamma^i_{ii} = \partial_i\left(\ln\sqrt{|g_{ii}|}\right) \Rightarrow \Gamma^r_{rr} = \Gamma^\theta_{\theta\theta} = \Gamma^\phi_{\phi\phi} = 0$

All other components are automatically zero.

a) $\nabla_i \psi = \partial_i \psi = \left(\frac{\partial\psi}{\partial r}, \frac{\partial\psi}{\partial\theta}, \frac{\partial\psi}{\partial\phi}\right)$

covariant derivative reduces to partial on scalars

This probably disagrees w/ what you see in E+M texts so let's look closer.

If we include the basis dual vectors and write the full expression:

$$\nabla\psi = \partial\psi = \frac{\partial\psi}{\partial r} e^{(r)} + \frac{\partial\psi}{\partial\theta} e^{(\theta)} + \frac{\partial\psi}{\partial\phi} e^{(\phi)}$$

What is important is that $e^{(r)}, e^{(\theta)}, e^{(\phi)}$ are coordinate dual basis vectors. However most E+M books use orthonormal dual basis vectors $e^{(\hat{r})}, e^{(\hat{\theta})}, e^{(\hat{\phi})}$.

This means: $e^{(\hat{r})} \cdot e^{(\hat{r})} = 1$ but for ours: $e^{(r)} \cdot e^{(r)} = g^{rr} = 1$

$e^{(\hat{\theta})} \cdot e^{(\hat{\theta})} = 1$ $\qquad e^{(\theta)} \cdot e^{(\theta)} = g^{\theta\theta} = r^{-2}$

$e^{(\hat{\phi})} \cdot e^{(\hat{\phi})} = 1$ $\qquad e^{(\phi)} \cdot e^{(\phi)} = g^{\phi\phi} = \frac{1}{r^2 \sin^2\theta}$

The above implies: $e^{(r)} = e^{(\hat{r})}$ $\qquad e^{(\hat{r})} = e^{(r)}$

$e^{(\theta)} = \frac{1}{r} e^{(\hat{\theta})}$ $\quad \Rightarrow \quad e^{(\hat{\theta})} = r e^{(\theta)}$

$e^{(\phi)} = \frac{1}{r\sin\theta} e^{(\hat{\phi})}$ $\qquad e^{(\hat{\phi})} = r\sin\theta\, e^{(\phi)}$

Now we must have that:

$$\nabla\psi = (\nabla\psi)_{\hat{r}} e^{(\hat{r})} + (\nabla\psi)_{\hat{\theta}} e^{(\hat{\theta})} + (\nabla\psi)_{\hat{\phi}} e^{(\hat{\phi})} = \frac{\partial\psi}{\partial r} e^{(r)} + \frac{\partial\psi}{\partial\theta} e^{(\theta)} + \frac{\partial\psi}{\partial\phi} e^{(\phi)}$$

Substituting the expressions for $e^{(r)}, e^{(\theta)}, e^{(\phi)}$ in terms of $e^{(\hat{r})}, e^{(\hat{\theta})}, e^{(\hat{\phi})}$:

$$\nabla\psi = \underbrace{\frac{\partial\psi}{\partial r}}_{(\nabla\psi)_{\hat{r}}} e^{(\hat{r})} + \underbrace{\frac{\partial\psi}{\partial\theta}\frac{1}{r}}_{(\nabla\psi)_{\hat{\theta}}} e^{(\hat{\theta})} + \underbrace{\frac{\partial\psi}{\partial\phi}\frac{1}{r\sin\theta}}_{(\nabla\psi)_{\hat{\phi}}} e^{(\hat{\phi})}$$

Which should agree w/ most E+M books

b. $\nabla \cdot V = \nabla_i V^i = \partial_i V^i + \Gamma^i_{ij} V^j$

$$= \frac{\partial V^r}{\partial r} + \frac{\partial V^\theta}{\partial \theta} + \frac{\partial V^\phi}{\partial \phi} + \Gamma^\theta_{\theta r} V^r + \Gamma^\phi_{\phi r} V^r + \Gamma^\phi_{\phi \theta} V^\theta$$

$$= \frac{\partial V^r}{\partial r} + \frac{\partial V^\theta}{\partial \theta} + \frac{\partial V^\phi}{\partial \phi} + \frac{2}{r} V^r + \cot\theta \, V^\theta$$

Again this does not agree w/ E+M texts, but we can convert our coordinate basis expression to an orthonormal one. Using a similar argument to the above:

$$e_{(\hat{r})} \cdot e_{(\hat{r})} = 1 \qquad e_{(r)} \cdot e_{(r)} = g_{rr} = 1$$

$$e_{(\hat{\theta})} \cdot e_{(\hat{\theta})} = 1 \quad \text{and} \quad e_{(\theta)} \cdot e_{(\theta)} = g_{\theta\theta} = r^2$$

$$e_{(\hat{\phi})} \cdot e_{(\hat{\phi})} = 1 \qquad e_{(\phi)} \cdot e_{(\phi)} = g_{\phi\phi} = r^2 \sin^2\theta$$

$$e_{(r)} = e_{(\hat{r})} \qquad e_{(\hat{r})} = e_{(r)}$$

$$e_{(\theta)} = r \, e_{(\hat{\theta})} \qquad e_{(\hat{\theta})} = \frac{1}{r} e_{(\theta)}$$

$$e_{(\phi)} = r \sin\theta \, e_{(\hat{\phi})} \qquad e_{(\hat{\phi})} = \frac{1}{r\sin\theta} e_{(\phi)}$$

Then using that:

$$V = V^r e_{(r)} + V^\theta e_{(\theta)} + V^\phi e_{(\phi)} = \underbrace{V^{\hat{r}} e_{(\hat{r})} + V^{\hat{\theta}} e_{(\hat{\theta})} + V^{\hat{\phi}} e_{(\hat{\phi})}}_{V^{\hat{r}} e_{(r)} + V^{\hat{\theta}} \frac{1}{r} e_{(\theta)} + V^{\hat{\phi}} \frac{1}{r\sin\theta} e_{(\phi)}}$$

And our expression above becomes:

$$\nabla \cdot V = \frac{\partial V^{\hat{r}}}{\partial r} + \frac{\partial}{\partial \theta}\left(\frac{V^{\hat{\theta}}}{r}\right) + \frac{\partial}{\partial \phi}\left(\frac{V^{\hat{\phi}}}{r\sin\theta}\right) + \frac{2}{r} V^{\hat{r}} + \frac{\cot\theta}{r} V^{\hat{\theta}}$$

In agreement w/ E+M texts

4. Consider the vector field in $\mathbb{R}^3$ that corresponds to the electric field of a unit point charge at the origin in spherical polar coordinates. You may ignore time in this problem. Find expressions in spherical polar coordinates for:
   a) The covariant derivative of the electric field.
   b) The directional covariant derivative of the electric field along the path given by $x^\mu(\lambda) = \big(r(\lambda), \theta(\lambda), \phi(\lambda)\big) = (\lambda^2, \frac{\pi}{2}, \lambda)$.

Let's first collect the Christoffel symbols which we computed in the last assignment:

$$g_{rr} = 1 \qquad g_{\theta\theta} = r^2 \qquad g_{\phi\phi} = r^2 \sin^2\theta$$

$$\partial_r g_{\theta\theta} = 2r \qquad \partial_r g_{\phi\phi} = 2r \sin^2\theta$$

$$\partial_\theta g_{\phi\phi} = 2r^2 \sin\theta\cos\theta$$

Then using $i, j, k \in \{r, \theta, \phi\}$ we get:

$$\Gamma^i_{jk} = 0 \Rightarrow \Gamma^r_{\theta\phi} = \Gamma^\theta_{r\phi} = \Gamma^\phi_{\theta r} = 0 \quad \text{or any switch of lower indices}$$

$$\Gamma^i_{jj} = -\frac{1}{2}\frac{1}{g_{ii}}\partial_i g_{jj} \Rightarrow \Gamma^r_{\theta\theta} = -r, \; \Gamma^r_{\phi\phi} = -r\sin^2\theta, \; \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^i_{ji} = \partial_j\left(\ln\sqrt{|g_{ii}|}\right) \Rightarrow \Gamma^\theta_{r\theta} = \frac{1}{r}, \; \Gamma^\phi_{r\phi} = \frac{1}{r}, \; \Gamma^\phi_{\theta\phi} = \cot\theta$$

$$\Gamma^i_{ii} = \partial_i\left(\ln\sqrt{|g_{ii}|}\right) \Rightarrow \Gamma^r_{rr} = \Gamma^\theta_{\theta\theta} = \Gamma^\phi_{\phi\phi} = 0$$

All other components are automatically zero.

a) $\vec{E} = \frac{kq}{r^2}\hat{r} \Rightarrow E^r = \frac{kq}{r^2}, \; E^\theta = 0, \; E^\phi = 0$

$$\nabla_\mu E^\nu = \partial_\mu E^\nu + \Gamma^\nu_{\mu\lambda} E^\lambda$$

This is a 9 component expression.

Note: Not the divergence!

Since only $E^r \neq 0$ the only term in this sum will be $\lambda = r$

$\mu = r, \nu = r \quad \nabla_r E^r = \partial_r E^r + \cancel{\Gamma^r_{r\lambda} E^\lambda} = -\frac{2kq}{r^3}$

$\mu = r, \nu = \theta \quad \nabla_r E^\theta = \cancel{\partial_r E^\theta} + \cancel{\Gamma^\theta_{r\lambda} E^\lambda} = 0$

$\mu = r, \nu = \phi \quad \nabla_r E^\phi = \cancel{\partial_r E^\phi} + \cancel{\Gamma^\phi_{r\lambda} E^\lambda} = 0$

$\mu = \theta, \nu = r \quad \nabla_\theta E^r = \cancel{\partial_\theta E^r} + \cancel{\Gamma^r_{\theta\lambda} E^\lambda} = 0$

$\mu = \theta, \nu = \theta \quad \nabla_\theta E^\theta = \cancel{\partial_\theta E^\theta} + \Gamma^\theta_{\theta\lambda} E^\lambda = \frac{1}{r}\frac{kq}{r^2} = \frac{kq}{r^3}$

$\mu = \theta, \nu = \phi \quad \nabla_\theta E^\phi = \cancel{\partial_\theta E^\phi} + \cancel{\Gamma^\phi_{\theta\lambda} E^\lambda} = 0$

$\mu = \phi, \nu = r \quad \nabla_\phi E^r = \cancel{\partial_\phi E^r} + \cancel{\Gamma^r_{\phi\lambda} E^\lambda} = 0$

$\mu = \phi, \nu = \theta \quad \nabla_\phi E^\theta = \cancel{\partial_\phi E^\theta} + \cancel{\Gamma^\theta_{\phi\lambda} E^\lambda} = 0$

$\mu = \phi, \nu = \phi \quad \nabla_\phi E^\phi = \cancel{\partial_\phi E^\phi} + \Gamma^\phi_{\phi\lambda} E^\lambda = \frac{1}{r}\frac{kq}{r^2} = \frac{kq}{r^3}$

Note that if we had computed the divergence, i.e. $\nabla_\mu E^\mu = \nabla_r E^r + \nabla_\theta E^\theta + \nabla_\phi E^\phi$ we would have gotten as expected.

b) For $x^\mu(\lambda) = (\lambda^2, \frac{\pi}{2}, \lambda)$ we have:

$$\frac{DE^\nu}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu E^\nu = 2\lambda \nabla_r E^\nu + \nabla_\phi E^\nu = \Big(\overbrace{2\lambda\left(-\frac{2kq}{r^3}\right)}^{\nu=r}, \overbrace{0}^{\nu=\theta}, \overbrace{\quad}^{\nu=\phi}$$

$$= \left(-\frac{4kq}{\lambda^5}, 0, \frac{kq}{\lambda^6}\right)$$

5. Consider a unit 2-sphere with coordinates (θ, ϕ) and metric $ds^2 = d\theta^2 + sin^2\theta d\phi^2$.
   a) Take a vector with components $V^\mu = (1,0)$ and parallel transport it once around a circle of constant latitude. What are the components of the resulting vector as a function of the polar angle θ of the circle of constant latitude.
   b) Show that lines of constant longitude (ϕ = constant) are geodesics, and that the only line of constant latitude (θ = constant) that is a geodesic is the equator ($\theta = \frac{\pi}{2}$).

From ds^2 we find $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \Rightarrow g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix} \Rightarrow \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta$, $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$ (all other $\Gamma = 0$)

a) To $/\!/$-transport $V^\mu = (1,0)$ around a line of constant latitude, i.e. $\underbrace{x^\mu(\theta,\lambda) = (\theta,\lambda)}_{\text{parameterization}}$ (θ fixed) we solve:

$$\frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}V^\rho = 0 \quad w/ \quad V^\mu(\lambda=0) = (1,0)$$

$$\theta: \frac{dV^\theta}{d\lambda} + \Gamma^\theta_{\phi\phi}\frac{dx^\phi}{d\lambda}V^\phi = \frac{dV^\theta}{d\lambda} - \sin\theta\cos\theta\, V^\phi = 0$$

$$\phi: \frac{dV^\phi}{d\lambda} + \Gamma^\phi_{\phi\theta}\frac{dx^\phi}{d\lambda}V^\theta = \frac{dV^\phi}{d\lambda} + \cot\theta\, V^\theta = 0$$

Taking an additional $\frac{d}{d\lambda}$ of these 2 equations gives:

$$\theta: \frac{d^2V^\theta}{d\lambda^2} - \sin\theta\cos\theta\frac{dV^\phi}{d\lambda} = 0 = \frac{d^2V^\theta}{d\lambda^2} - \sin\theta\cos\theta(-\cot\theta V^\theta) = \frac{d^2V^\theta}{d\lambda^2} + \cos^2\theta\, V^\theta \Rightarrow \frac{d^2V^\theta}{d\lambda^2} = -k^2V^\theta \quad k \equiv \cos\theta$$

$$\phi: \frac{d^2V^\phi}{d\lambda^2} + \cot\theta\frac{dV^\theta}{d\lambda} = 0 = \frac{d^2V^\phi}{d\lambda^2} + \cot\theta(\sin\theta\cos\theta\, V^\phi) = \frac{d^2V^\phi}{d\lambda^2} + \cos^2\theta\, V^\phi \Rightarrow \frac{d^2V^\phi}{d\lambda^2} = -k^2V^\phi$$

The general solutions are:

$V^\theta(\lambda) = A\sin(k\lambda) + B\cos(k\lambda)$

$V^\phi(\lambda) = C\sin(k\lambda) + D\cos(k\lambda)$

$\Longrightarrow$ Applying initial conditions

$V^\theta(\lambda=0) = 1 \Rightarrow V^\theta(\lambda) = A\sin(k\lambda) + \cos(k\lambda)$

$V^\phi(\lambda=0) = 0 \Rightarrow V^\phi(\lambda) = C\sin(k\lambda)$

But they must still satisfy:

$\frac{dV^\theta}{d\lambda} - \sin\theta\cos\theta\, V^\phi = 0 \Rightarrow Ak\cos(k\lambda) - k\sin(k\lambda) - \sin\theta\cos\theta\, C\sin(k\lambda) = 0$

$\frac{dV^\phi}{d\lambda} + \cot\theta\, V^\theta = 0 \Rightarrow Ck\cos(k\lambda) + \cot\theta A\sin(k\lambda) + \cot\theta\cos(k\lambda) = 0$

$\Bigg\}$ $A = 0$, $C = -\frac{k}{\sin\theta\cos\theta} = -\frac{1}{\sin\theta}$

Then $V^\mu(\lambda) = \left(\cos(\lambda\cos\theta), -\frac{1}{\sin\theta}\sin(\lambda\cos\theta)\right)$

b) A line of constant longitude is $x^\mu(\lambda) = (\lambda, \underbrace{\phi}_{\text{constant}}) \Rightarrow \frac{d\theta}{d\lambda} = 1, \frac{d\phi}{d\lambda} = 0$

Then: $\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0 \Rightarrow \begin{cases} \theta: \frac{d^2\theta}{d\lambda^2} + \Gamma^\theta_{\phi\phi}\frac{d\phi}{d\lambda}\frac{d\phi}{d\lambda} = 0 + 0 = 0 \; ✓ \\ \phi: \frac{d^2\phi}{d\lambda^2} + \Gamma^\phi_{\theta\phi}\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} + \Gamma^\phi_{\phi\theta}\frac{d\phi}{d\lambda}\frac{d\theta}{d\lambda} = 0 + 0 + 0 = 0 \; ✓ \end{cases}$

A line of constant latitude is $x^\mu(\lambda) = (\underbrace{\theta}_{\text{constant}}, \lambda) \Rightarrow \frac{d\theta}{d\lambda} = 0, \frac{d\phi}{d\lambda} = 1$

Then: $\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0 \Rightarrow \begin{cases} \theta: \frac{d^2\theta}{d\lambda^2} + \Gamma^\theta_{\phi\phi}\frac{d\phi}{d\lambda}\frac{d\phi}{d\lambda} = 0 - \sin\theta\cos\theta = 0 \text{ if } \theta = 0, \frac{\pi}{2}, \pi \\ \phi: \frac{d^2\phi}{d\lambda^2} + \Gamma^\phi_{\theta\phi}\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} + \Gamma^\phi_{\phi\theta}\frac{d\phi}{d\lambda}\frac{d\theta}{d\lambda} = 0 + 0 + 0 \; ✓ \end{cases}$

So lines of constant latitude are only geodesic if we are at the north/south poles or the equator.

6. Show that extremizing the functional $\int ds$ for the space $\mathbb{R}^2$ in polar coordinates (r, θ) leads to a set of differential equations for $r(\lambda)$ and $\theta(\lambda)$ that are the same as the geodesic equations. You can simply begin with the usual result of extremizing an action, i.e. the Euler-Lagrange equations.

We have: $L = \sqrt{v_r^2 + r^2 v_\theta^2}$ where $v_r = \frac{dr}{ds}$, $v_\theta = \frac{d\theta}{ds}$ and we can use that $\frac{dL}{ds} = 0$

The Euler-Lagrange equations $\frac{d}{ds}\left(\frac{\partial L}{\partial v_i}\right) - \frac{\partial L}{\partial x^i} = 0$ become:

$$r: \frac{d}{ds}\left(\frac{\partial L}{\partial v_r}\right) - \frac{\partial L}{\partial r} = \frac{1}{2}\frac{d}{ds}\left(\frac{2v_r}{L}\right) - \frac{1}{2}\frac{2 r v_\theta^2}{L} = \frac{1}{L}\left[\frac{d^2r}{ds^2} - r\left(\frac{d\theta}{ds}\right)^2\right] = 0 \Rightarrow \frac{d^2r}{ds^2} - r\left(\frac{d\theta}{ds}\right)^2 = 0$$

$$\theta: \frac{d}{ds}\left(\frac{\partial L}{\partial v_\theta}\right) - \frac{\partial L}{\partial \theta} = \frac{1}{2}\frac{d}{ds}\left(\frac{2r^2 v_\theta}{L}\right) = \frac{r^2}{L}\left[\frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} + \frac{d^2\theta}{ds^2}\right] = 0 \Rightarrow \frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} = 0$$