General Relativity HW9 Problems

1. Find explicit expressions for all of the Killing vectors K^μ for 1+3D Minkowski spcae M^4. Be careful to recall that what appears in Killing's equation are the dual Killing vectors!

$\mathbb{M}^4$ w/ (t,x,y,z) $\quad ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

All Γ's $= 0$.

Then $\nabla_{(\mu}K_{\nu)}$ becomes:

$$\begin{array}{lll} \partial_t K_t = 0 & \partial_t K_x + \partial_x K_t = 0 & \partial_x K_y + \partial_y K_x = 0 \\ \partial_x K_x = 0 & \partial_t K_y + \partial_y K_t = 0 & \partial_x K_z + \partial_z K_x = 0 \\ \partial_y K_y = 0 & \partial_t K_z + \partial_z K_t = 0 & \partial_y K_z + \partial_z K_y = 0 \\ \partial_z K_z = 0 & & \end{array}$$

All 10 of these must be satisfied by any solution.

Since $\mathbb{M}^4$ is maximally symmetric we expect $\frac{1}{2}4(4+1) = 10$ solutions.

For each solution K_μ we find the corresponding Killing vector $K^\mu = g^{\mu\nu}K_\nu$ and the conserved dual momentum $P_{6*} = K^\mu P_\mu$ where $P_\mu = (-E, P_x, P_y, P_z)$.

4 easy ones are:

$$\left.\begin{array}{lll} K_\mu = (1,0,0,0) \Rightarrow K^\mu = (-1,0,0,0) & \Rightarrow & K^\mu P_\mu = E \\ K_\mu = (0,1,0,0) \Rightarrow K^\mu = (0,1,0,0) & \Rightarrow & K^\mu P_\mu = P_x \\ K_\mu = (0,0,1,0) \Rightarrow K^\mu = (0,0,1,0) & \Rightarrow & K^\mu P_\mu = P_y \\ K_\mu = (0,0,0,1) \Rightarrow K^\mu = (0,0,0,1) & \Rightarrow & K^\mu P_\mu = P_z \end{array}\right\} \text{Expected since } g_{\mu\nu} \text{ does not depend on } t,x,y,z!$$

3 less obvious ones are:

$K_\mu = (0,-y,x,0) \Rightarrow K^\mu = (0,-y,x,0) \Rightarrow K^\mu P_\mu = -yP_x + xP_y = $ "L_z"

$K_\mu = (0,z,0,-x) \Rightarrow K^\mu = (0,z,0,-x) \Rightarrow K^\mu P_\mu = zP_x - xP_z = $ "L_y"

$K_\mu = (0,0,-z,y) \Rightarrow K^\mu = (0,0,-z,y) \Rightarrow K^\mu P_\mu = yP_z - zP_y = $ "L_x"

These 3 are expected since $\mathbb{M}^4$ is spatially isotropic (rotationally invariant).

Finally 3 less intuitive solutions are:

$$\left.\begin{aligned} K_\mu &= (-x, t, 0, 0) \Rightarrow K^\mu = (x, t, 0, 0) \Rightarrow K^\mu P_\mu = xE + tP_x \\ K_\mu &= (-y, 0, t, 0) \Rightarrow K^\mu = (y, 0, t, 0) \Rightarrow K^\mu P_\mu = yE + tP_y \\ K_\mu &= (-z, 0, 0, t) \Rightarrow K^\mu = (z, 0, 0, t) \Rightarrow K^\mu P_\mu = zE + tP_z \end{aligned}\right\} \text{Strange, but true!}$$

Problems 2,3 and 4, see Mathematica notebook for solutions.