General Relativity HW9 Problems

1. Once across the event horizon of a Schwarzschild black hole of mass M, what is the longest proper time an observer can spend before reaching the singularity? Hint: You should ignore any attempts at angular motion and try to find an expression for $\frac{dr}{d\tau}$ where $d\tau^2 = -ds^2$.

It would be useful to determine the rate $\frac{dr}{d\tau}$ before proceeding.

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2\cancel{d\Omega^2}^{\,0} = -d\tau^2$$

Then: $\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 = 1$

$$\frac{dr}{d\tau} = \pm\sqrt{\left(\frac{2GM}{r}-1\right) + \underbrace{\left(1-\frac{2GM}{r}\right)^2\left(\frac{dt}{d\tau}\right)^2}}$$

This term is ≥ 0 so the smallest rate occurs when this $= 0$. The $(-)$ root is for decreasing r!

$$\frac{dr}{d\tau} = -\sqrt{\left(\frac{2GM}{r}-1\right)}$$

Or: $d\tau_{max} = \frac{-dr}{\sqrt{\left(\frac{2GM}{r}-1\right)}} \Rightarrow \tau_{max} = \int_{2GM}^{0}\frac{-dr}{\sqrt{\frac{2GM}{r}-1}} = \left[r\sqrt{-1+\frac{2GM}{r}} + GM\tan^{-1}\left(\frac{\sqrt{\frac{2GM}{r}-1}\,(r-GM)}{r-2GM}\right)\right]_{2GM}^{0}$

$$= GM\tan^{-1}(\infty) - GM\tan^{-1}(-\infty)$$

$$= \boxed{GM\pi}$$

2. Consider the spacetime specified by the line element

$$ds^2 = -(1-\frac{GM}{r})^2 dt^2 + (1-\frac{GM}{r})^{-2} dr^2 + r^2 d\Omega^2$$

a) Find a transformation to Eddington-Finkelstein-like coordinates (v,r,θ,ϕ) such that $g_{rr} = 0$ and show that the geometry is not singular at $r = GM$. You may start by looking for a new time coordinate $t = v + g(r)$ with $g(r)$ such that the g_{rr} term of the metric in the new coordinates is 0.

b) Sketch a plot analogous to our picture in class (EF for Schwarzschild) of the light cones in this geometry. That is take $ds^2 = 0$ with $d\Omega = 0$ and look for solutions from the metric.

a) We want a new "time" coordinate v s.t. $g_{rr} = 0$.

Assuming a form $t = v + g(r) \Rightarrow dt = dv + \frac{\partial g}{\partial r} dr$

Inserting this into the metric:

$$ds^2 = -(1-\frac{GM}{r})^2 (dv + \frac{\partial g}{\partial r} dr)^2 + (1-\frac{GM}{r})^{-2} dr^2 + r^2 d\Omega^2$$

Expanding and collecting all dr^2 coefficients we find:

$$g_{rr} = -(1-\frac{GM}{r})^2 (\frac{\partial g}{\partial r})^2 + (1-\frac{GM}{r})^{-2} = 0 \Rightarrow \frac{\partial g}{\partial r} = \pm(1-\frac{GM}{r})^{-2}$$

I chose the negative root (you can use either) and feeding to Mathematica yields: $g(r) = -r - 2GM \ln|r-GM| + \frac{G^2M^2}{r-GM}$

$$t = v - r - 2GM\ln|r-GM| - \frac{G^2M^2}{r-GM}$$

$$dt = dv - dr - \frac{2GM\,dr}{r-GM} - \frac{G^2M^2 dr}{(r-GM)^2}$$

$$= dv - (1 + \frac{2GM}{r-GM} + \frac{G^2M^2}{(r-GM)^2}) dr$$

$$= dv - (r^2 - 2GMr + G^2M^2 + 2GMr - 2G^2M^2 + G^2M^2) \frac{dr}{(r-GM)^2}$$

$$= dv - \frac{r^2 dr}{(r-GM)^2} = dv - (1-\frac{GM}{r})^{-2} dr$$

$$dt^2 = dv^2 + (1-\frac{GM}{r})^{-4} dr^2 - 2\frac{dv\,dr}{(1-GM/r)^2}$$

$$ds^2 = -(1-\frac{GM}{r})^2 dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$ which is non-singular at $r = GM$!

b) For radial null geodesics ($ds^2 = 0$ w/ $d\Omega = 0$):

$$0 = -\left(1 - \frac{GM}{r}\right)^2 dv^2 + 2\,dv\,dr$$

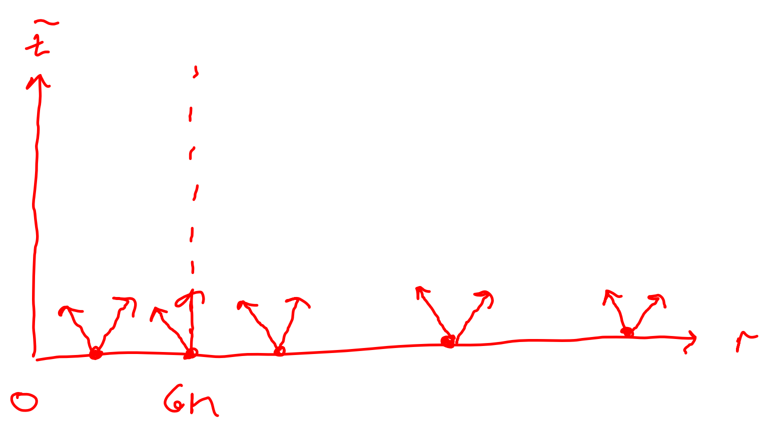
Solutions include:

i) $dv = 0 \Rightarrow dt = -\left(1 - \frac{GM}{r}\right)^{-2} dr$

$$\frac{dr}{dt} = -\left(1 - \frac{GM}{r}\right)^2 \leq 0 \quad \text{for } \underline{\underline{\text{all}}}\ r \ \text{(ingoing)}$$

ii) $\frac{dr}{dv} = \frac{1}{2}\left(1 - \frac{GM}{r}\right)^2 \geq 0$ for $\underline{\underline{\text{all}}}$ r (outgoing) Different than the Schwarzchild case!!

iii) $dr = 0$ w/ $r = GM$ (fixed at $r = GM$)



3. Two observers in two rockets are hovering above a Schwarzschild black hole of mass *M*. They hover at a fixed radius *r* such that

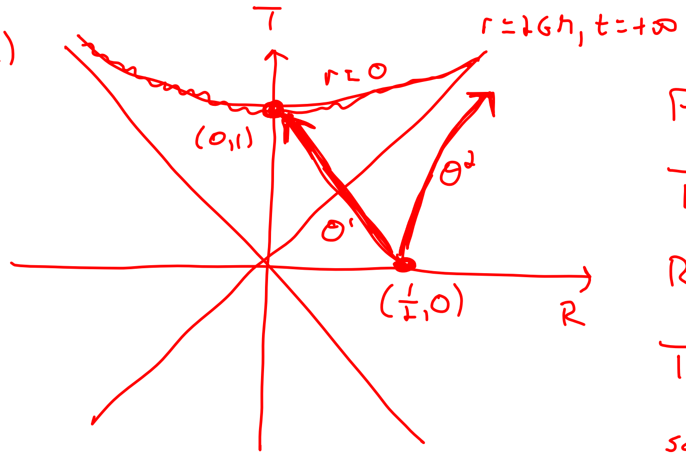
$$\left(\frac{r}{2GM}-1\right)^{1/2} e^{r/4GM} = 1/2$$

with fixed angular position. The first observer leaves this position at *t = 0* and travels into the black hole **on a straight line path in a Kruskal diagram** until destroyed in the singularity at the point where the singularity crosses the line *R = 0* where *R* is the Kruskal radial coordinate (Note *r,t* are Schwarzschild coordinate values). The other observer continues to hover at radius *r*.

a) On a Kruskal diagram, sketch the worldlines of the two observers.

b) Is the observer who goes into the black hole following a timelike worldline?

a)



For $t=0$ and r where $\left(\frac{r}{2GM}-1\right)^{1/2} e^{\frac{r}{4GM}} = \frac{1}{2}$

$$T_i = \left(\frac{r}{2GM}-1\right)^{1/2} e^{\frac{r}{4GM}} \sinh\left(\frac{t}{4GM}\right) = \boxed{0}$$

$$R_i = \left(\frac{r}{2GM}-1\right)^{1/2} e^{\frac{r}{4GM}} \cosh\left(\frac{t}{4GM}\right) = \boxed{\frac{1}{2}}$$

The $r=0$ singularity is at $T^2-R^2=1$

so when $R_f=0 \Rightarrow \boxed{T_f=1}$

b) For the infalling observer $m = \frac{T_f-T_i}{R_f-R_i} = \frac{1-0}{0-\frac{1}{2}} = 2 > \underbrace{1}_{\text{slope of light cone}}$ so

yes they are on a time-like geodesic.

4. Consider the Reissner-Nordstrom metric for a black hole with net charge Q:

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)dt^2 + \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega^2$$

a) Notice that the metric is badly behaved when $\Delta \to 0$ and when $\Delta \to \infty$. Find the values of r for which these occur.

b) Prove that $\Delta \to \infty$ represents a true curvature singularity by calculating the curvature related invariant $R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}$.

c) What is the energy-momentum tensor associated with this geometry? Compare this to the energy-momentum tensor associated to the Kerr geometry. Why are they so different? Hint: Use Mathematica for both of these!

d) What is the form of the metric for the "extremal" case when $G^2M^2 = GQ^2$? Notice anything?

a) $\Delta \to \infty$ when $r \to 0$ since $\frac{2GM}{r}$ and $\frac{GQ^2}{r^2} \to \infty$.

$\Delta \to 0$ when $1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} = 0 \Rightarrow r^2 - 2GMr + GQ^2 = 0$

$$r_\pm = \frac{2GM \pm \sqrt{4G^2M^2 - 4GQ^2}}{2}$$

$$r_\pm = GM \pm \sqrt{G^2M^2 - GQ^2}$$

b) See Mathematica notebook to find $R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} \propto \frac{1}{r^{14}}$

so when $r \to 0$, this curvature invariant $\to \infty$.

c) Using that $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, we used Mathematica to calculate $G_{\mu\nu}$ (see notebook) giving:

$$T^{(R\text{-}N)}_{\mu\nu} = \frac{1}{8\pi G}\begin{pmatrix} \frac{G^2Q^2}{r^6}(G^2Q^2 - 2GMr + r^2) & 0 & 0 & 0 \\ 0 & \frac{G^2Q^2}{r^2}\frac{1}{(G^2Q^2 - 2GMr + r^2)} & 0 & 0 \\ 0 & 0 & \frac{G^2Q^2}{r^2} & 0 \\ 0 & 0 & 0 & \frac{G^2Q^2 \sin(\theta)^2}{r^2} \end{pmatrix}$$

but $T^{(K)}_{\mu\nu} = 0$. This is due to the fact that outside of a spinning black hole there is nothing to contribute to $T_{\mu\nu}$, but for a charged black hole, the electric field outside of it will!