Mathematical Methods in Physics HW2

1. Consider the linear transformation of D on the vector space P_3 with representative element $x = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3$. Find the matrix form of D with respect to the basis $\{1, t, t^2, t^3\}$. Now find the matrix form of the operator D^2 in two ways. First by considering the action of D^2 on x. And second by relating the matrix representation of D to the matrix representation of D^2.

2. This one is simple and straightforward, but I think the results are best absorbed if you just work out little examples of the results I gave you on determinants. Show the following:

a) A common factor in each row or column may be factored out, hence show that $detA = det\begin{pmatrix} 6a & 2b \\ 3c & d \end{pmatrix} = 6\ detB$ where $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

b) If any row or column is zero then $detA = 0$. Calculate $detA$ where $A = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$ and when $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$.

c) Interchanging two rows or columns changes the sign of the determinant. Calculate $detB$ when $B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$ and $detC$ when $C = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$ as well as $detD$ when $D = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$ and compare to $detA$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

d) If any two rows or columns are equal, then $detA = 0$. So evaluate $detA$ for $A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$ and $detB$ for $B = \begin{pmatrix} a & a \\ c & c \end{pmatrix}$.

e) Show explicitly that for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ then $\det(AB) = \det(A)\det(B)$.

f) A scalar multiple of a row or column may be added to another row or column without changing the determinant. Hence show that $detA = det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = det\begin{pmatrix} a+2c & b+2d \\ c & d \end{pmatrix} = det\begin{pmatrix} a+3b & b \\ c+3d & d \end{pmatrix} = det\begin{pmatrix} a+2c+3b+6d & b+2d \\ c+3d & d \end{pmatrix}$.

g) If the row or column "vectors" of a matrix are linearly dependent, then $detA = 0$. So evaluate $detA$ when $A = \begin{pmatrix} a & b \\ -3a & -3b \end{pmatrix}$.

3. Now things get a bit harder. These are proofs, not examples.

a) Use any of the earlier results (a-e) to prove the next to last result (f).

b) Use any of the earlier results (a-f) to prove the last result (g).

4. Consider a linear transformation which takes vectors in $\mathbb{R}^3$ and projects them onto a plane defined by the axis through $x = y = z = 1$ and the origin. Even though this is a projection, an inner product will not be necessary in solving it.

a) Write down the matrix version of this linear transformation. And give an example of acting upon a vector with this operator to create the projected version of the vector. Don't use the trivial cases, i.e. a vector along the axis or a vector already in the plane.

b) What is the determinant of your linear transformation as a matrix?

Now, imagine a change of basis which takes the axis defining the plane and aligns it with the x-axis.

c) Start by finding the operator matrix which takes vector components into their new form.

d) Now find the transformed version of the projection operator.

e) Verify that the transformed version of this story matches the pre-transformed version.

5. Evaluate the classical adjoint of $M = \begin{pmatrix} a & b & c \\ d & e & f \\ h & i & j \end{pmatrix}$.

6. Supposing that M^{-1} in the previous question exists, calculate it.

7. We found in an example in class that a rotation in the xy-plane in $\mathbb{R}^3$ has three distinct eigenvalues and three distinct eigenvectors. Prove that the same is true for any rotation in $\mathbb{R}^3$.

8. Is the following matrix diagonalizable via a similarity transformation? $M = \begin{pmatrix} 3 & 0 & 0 & -1 \\ -\frac{3}{\sqrt{2}} & 2 & 0 & -\frac{3}{\sqrt{2}} \\ 0 & 0 & 4 & 0 \\ -1 & 0 & 0 & 3 \end{pmatrix}$

Explain your reasoning.