Mathematical Methods in Physics HW6

1. Consider the matrix $A_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Now consider a perturbation given by

$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. For the new matrix, determine the eigenvalues to first order in ϵ and the appropriate zeroth order eigenvectors for corrections in ϵ using degenerate perturbation theory.

2. Check that your results from problem (1) for the eigenvalues agree with what you would obtain by directly determining the eigenvalues as an expansion in ϵ.

3. Consider the function $f(x) = \begin{cases} 1 & \text{for irrational } x \\ 0 & \text{for rational } x \end{cases}$ on the closed domain $x \in [0,1]$ with inner product $(f_1, f_2) = \int_0^1 f_1^* f_2 dx$. Find $\|f\|^2$ using both the Riemannian measure as well as the Lebesgue measure. Do they agree?

4. a) For the example in class of a space which is not complete, i.e. continuous functions with norm defined by $\|x\| = \int_0^1 |x(t)| dt$, show that for the given sequence $x_n(t) = \begin{cases} 0 & 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ 1 + nt - \frac{n}{2} & \frac{1}{2} - \frac{1}{n} \le t \le \frac{1}{2} \\ 1 & \frac{1}{2} \le t \le 1 \end{cases}$,

$\|x_n - x_m\| = \frac{1}{2}\left|\frac{1}{n} - \frac{1}{m}\right|$.

b) Now imagine that the sequence was instead in a space which has as its norm $\|x\| = \max_{t \in [0,1]} |x(t)|$. In this case, prove that the sequence is not Cauchy.

5. Show that the sequence $g_n(x) = \frac{\cos(nx)}{\sqrt{n}}$ converges uniformly to $g(x) = 0$ for $x \in \mathbb{R}$. Does it converge pointwise?

6. Show that the sequence $f_n(x) = x^n$ on $x \in [0,1]$, converges pointwise but not uniformly. Does it converge in the mean?

7. Consider the function $g(x) = x^2 - x$ over the interval $x \in [0,1]$.

a) Construct the first two polynomials using the Weierstrass construction from class, i.e. calculate $P_1(x)$ and $P_2(x)$.
b) Construct the Taylor series for the function to fourth order?
c) Which one wins?
d) Consider instead the function given by $g(x) = \begin{cases} x & 0 \le x \le \frac{1}{2} \\ 1 - x & \frac{1}{2} \le x \le 1 \end{cases}$.

Which one wins in this case? Why?