Mathematical Methods in Physics HW7

1. Consider polynomials on the interval $x \in [0, \infty)$ with inner product $(f, g) = \int_0^\infty f^*(x) g(x) e^{-x}\, dx$.

a) Gram-Schmidt the linearly independent set $\{x^0, x^1, x^2, \dots\}$ to find the first three orthonormal polynomials $\hat{L}_n(x)$.

b) Make the first three orthogonal polynomials with the Rodrigues formula $L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}(e^{-x} x^n)$. What are the normalization factors?

c) Now consider the generating function $\varphi(x, t) = \frac{1}{1-t} e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x)$. Following some of the steps that we did in class (differentiating this w.r.t. to x and t then peeling off expressions that result from grabbing everything in front of t^n), you should generate two equations that can then be manipulated to derive the differential equation for which the Laguerre polynomials are solutions, i.e. $x L_n''(x) + (1 - x) L_n'(x) + n L_n(x) = 0$. Try this. Definitely get the two equations mentioned, and you can try to combine them to get the differential equation, but it can be quite tricky.

2. Consider starting with an operator $L = (\alpha_o x^2 + \alpha_1 x + \alpha_2) \frac{d^2}{dx^2} + (\beta_0 x + \beta_1) \frac{d}{dx}$. Use the conditions 1-3 from class to reduce this to a form such that $\alpha_0 \neq 0$ and the interval over which this is Hermitian is $[a, b]$. Determine the weight that would be used in the inner product in this case.

3. Find an expression for the Fourier transform of a product of three transformable functions $f_1(y), f_2(y), f_3(y)$ in terms of their transforms $g_1(k), g_2(k), g_3(k)$.

4. Using the result of problem 1, evaluate the Fourier transform of the product of the three functions $f_1(y) = e^{ik_1 y}$, $f_2(y) = e^{ik_2 y}$ and $f_3(y) = e^{ik_3 y}$. Verify that your answer makes sense by considering the product as a single function.

5. Find Y_{20} by Gram-Schmidting.

6. Find Y_{32} using whatever method you want.

7. Evaluate the Stieltjes integral of $f(x) = e^{-x}$ with the measure inducing function given by $g(x) = n$ for $n \leq x < n + 1$ where $n \in \mathbb{Z}$ (integers that is) over $x \in [0, \infty]$.

8. Verify that the resolution of the identity for the position operator X, i.e. $Xf(x) = xf(x)$, is given by $E(\mu) f(x) = \begin{cases} f(x) & x \leq \mu \\ 0 & x > \mu \end{cases}$.

9. Given that $E_A(\lambda)$ is the resolution of the identity for the operator A. Show that $E_A(\lambda_2) - E_A(\lambda_1) = [E_A(\lambda_2) - E_A(\lambda_1)]^2$ for $\lambda_2 > \lambda_1$, but not for $\lambda_1 > \lambda_2$.

10. Consider the quantum-mechanical observables L_x, L_y and L_z, which are represented as matrices as follows:

$$L_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The Hamiltonian for this system has the form $H = H_0 + L_z$ where H_0 commutes with L_x, L_y and L_z.

a) Supposer that at $t = 0$ we have prepared the system in an eigenstate of L_z, namely, the state belonging to the eigenvalues $m = +1$ of L_z. If we measure L_z at a later time $t = T$, what are the probabilities that we will find the values $+1$,0 or -1.

b) Suppose instead that we measure L_x at time $t = T$. What are the possible values of L_x that can be obtained and what are their probabilities?