Mathematical Methods in Physics HW9

1. Consider the functions $w_1(z) = \sqrt{z}$ and $w_2(z) = \sqrt{(z^2 - 1)}$. Recall that a branch point is a point in the z-plane about which if we follow the values of $w(z)$ along a closed contour encircling it, then when we complete the contour we come back to a different value. Note, this should include circles small enough to avoid also encircling other branch points, as well as infinitesimal circles which pinpoint the identity of the branch point. For example if $z = 0$ is the only branch point, then a circle of radius 1 around $z = 0$ includes more than just $z = 0$, but only $z = 0$ is the branch point.

a) Determine the branch points for each function. One of the things that you should consider is whether $R = \infty$ constitutes a branch point. In order to figure this out, you can use an inversion map $z \to \frac{1}{\alpha}$ and then consider the newly formed function of α and whether $\alpha = 0$ is a branch point. *Hint: For w_1 and w_2 there are only 2.*

b) Now construct the branch cuts and label them on a drawing of the z-plane (along with the branch points). The rule in selecting where the cuts can go is that any curve that tries to go around a single branch point must pass through a branch cut. Can you connect the points with cuts and achieve this?

c) Finally, instead of thinking of the z-plane, consider instead the Riemann sphere which is just the surface of a two-sphere to which we can map the z-plane if we consider $R = 0$ at the south pole and if we include the point $R = \infty$ at the north pole. Draw the points and cuts using this representation of the complex plane. Does this help you understand the results for part (b)?

2. Compute the Laurent expansion of $w(z) = \frac{z^{122}+3z^{41}+1}{z^{568}}$ around $z = 0$. Use the formula, and then check your answer as we did in class.

3. Compute the Laurent expansion of $w(z) = \frac{e^z}{z}$ around $z = 0$.

4. Evaluate the integral $I = \oint_C \frac{4z^3-1}{z(z-1)} dz$ around a contour centered around $z = 0$ and of radius $|z| = 5$.

5. Consider swapping the role of $z \leftrightarrow z^*$ in defining analytic functions. What needs to be changed in various definitions in order for the story to work out with t his swap? *Hint: Recall that analytic functions are normally only functions of z and have the lovely property that closed contour integrals in regions where they are analytic always give zero.*