Math Methods HW 5 Quiz   Name____________________

You can try both problems below, but you will only receive credit for the most correct solution.

. (*10 pts*) For the matrix $M = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, show that the standard construction of eigenvalues and eigenvectors agrees with the results from extremizing $I = (x, Mx)$ subject to $(x, x) = 1$.

$\det[M - \lambda I] = 0 = (-\lambda)^2 - 1 = 0 \;\Rightarrow\; \lambda_1 = +1, \lambda_2 = -1$

$Mx_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda_1 x_1 = \begin{pmatrix} a \\ b \end{pmatrix} \;\Rightarrow\; \begin{matrix} ib = a \\ -ia = b \end{matrix} \;\Rightarrow\; x_1 = \begin{pmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$Mx_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda_2 x_2 = \begin{pmatrix} -a \\ -b \end{pmatrix} \;\Rightarrow\; \begin{matrix} ib = -a \\ -ia = -b \end{matrix} \;\Rightarrow\; x_2 = \begin{pmatrix} \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$K = (x, Mx) - \lambda[(x, x) - 1] = (a^* \; b^*)\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} - \lambda(a^*a + b^*b - 1)$

$= ia^*b - ib^*a - \lambda a^*a + \lambda b^*b - \lambda$

$\frac{\delta K}{\delta a} = -ib^* - \lambda a^* = 0 \;\Rightarrow\; -ib^* = \lambda a^* \;\Rightarrow\; ib = \lambda a$

$\frac{\delta K}{\delta b} = ia^* - \lambda b^* = 0 \;\Rightarrow\; ia^* = \lambda b^* \;\Rightarrow\; -ia = \lambda b$

For $\lambda = +1 \;\Rightarrow\; ib = a, -ia = b \;\Rightarrow\; x_1 = \begin{pmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

For $\lambda = -1 \;\Rightarrow\; ib = -a, -ia = -b \;\Rightarrow\; x_2 = \begin{pmatrix} \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

2.(*10 pts*) Consider the matrix $A_0 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Now consider a perturbation given by

$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For the new matrix, determine the eigenvalues to first order in ϵ and the appropriate zeroth order eigenvectors for corrections in ϵ using degenerate perturbation theory.

Let's get the starting point: $det[A_0 - \lambda I] = 0 = (2-\lambda)(2-\lambda)$

$\lambda_{2,1}^{(0)} = 2, \;\; \lambda_{2,2}^{(0)} = 2$

For $\lambda_{2,i}^{(0)} = 2 \quad \Rightarrow \quad A_0 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix} \quad \Rightarrow \quad \hat{x}_{2,1}^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \;\; \hat{x}_{2,2}^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

For $\lambda_{2,i}^{(0)}$: $\quad M_{jk} = \left(\hat{x}_{2,j}^{(0)}, A_1 \hat{x}_{2,k}^{(0)}\right) \quad \Rightarrow \quad M = \begin{pmatrix} (\hat{x}_{2,1}^{(0)}, A_1 \hat{x}_{2,1}^{(0)}) & (\hat{x}_{2,1}^{(0)}, A_1 \hat{x}_{2,2}^{(0)}) \\ (\hat{x}_{2,2}^{(0)}, A_1 \hat{x}_{2,1}^{(0)}) & (\hat{x}_{2,2}^{(0)}, A_1 \hat{x}_{2,2}^{(0)}) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$det\left[M - \lambda_{2,i}^{(1)} I\right] = 0 = \left(-\lambda_{2,i}^{(1)}\right)^2 - 1 = {\lambda_{2,i}^{(1)}}^2 - 1 \quad \Rightarrow \quad \lambda_{2,1}^{(1)} = 1, \;\lambda_{2,2}^{(1)} = -1$

Therefore: $\lambda_{2,1} = 2 + \epsilon + \cdots, \;\lambda_{2,2} = 2 - \epsilon + \cdots$

$\lambda_{2,1}^{(1)}$: $M \begin{pmatrix} a \\ b \end{pmatrix} = \lambda_{2,1}^{(1)} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{matrix} b = a \\ a = b \end{matrix} \quad \Rightarrow \quad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = a_{1\,k}$

$\lambda_{2,2}^{(1)}$: $M \begin{pmatrix} a \\ b \end{pmatrix} = \lambda_{2,2}^{(1)} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{matrix} b = -a \\ a = -b \end{matrix} \quad \Rightarrow \quad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = a_{2\,k}$

$\hat{y}_{2,1}^{(0)} = \sum_k a_{1\,k}\, \hat{x}_{2,k}^{(0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$\hat{y}_{2,2}^{(0)} = \sum_k a_{2\,k}\, \hat{x}_{2,k}^{(0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$