Math Methods HW 6 Quiz Name___________________

You can try both problems below, but you will only receive credit for the most correct solution.

1. (*10 pts*) Consider the sequence of functions $f_n(x) = x^n$ on the interval $[0,1]$ with inner product $(f_n, f_m) = \int_0^1 f_n^* f_m dx$. Show that the sequence is Cauchy.

Consider $||f_n(x) - f_m(x)|| = \sqrt{\int_0^1 |f_n(x) - f_m(x)|^2 dt}$. Taking $n < m$ so that we can ignore the absolute value, this becomes: $||f_n(x) - f_m(x)|| = \sqrt{\int_0^1 [x^n - x^m] dt} = \sqrt{\frac{1}{n+1} - \frac{1}{m+1}}$.

For this to be Cauchy we want to show that: $||f_n(x) - f_m(x)|| < \epsilon$ for $n, m > N(\epsilon)$. Well, $\sqrt{\frac{1}{n+1} - \frac{1}{m+1}} < \epsilon \quad \Rightarrow \quad \frac{1}{n+1} - \frac{1}{m+1} < \epsilon^2$. If we fix n, then the largest value the left hand side can obtain is when $m \to \infty$. But the condition then becomes $\frac{1}{n+1} < \epsilon^2 \quad \Rightarrow \quad n > \frac{1}{\epsilon^2} - 1 = N(\epsilon)$. We can lower m to smaller values which will still satisfy this, but we can't bring $m < N(\epsilon)$, otherwise taking $n \to \infty$ will cause a problem. So overall, $n, m > \frac{1}{\epsilon^2} - 1$ guarantees that $||f_n(x) - f_m(x)|| < \epsilon$.

2. (*10 pts*) Consider the sequence $g_n(x) = \frac{x^n}{n}$ on the interval $[0,1]$ with inner product $(g_n, g_m) = \int_0^1 g_n^* g_m dx$. Does this converge pointwise, uniformly or in the mean?

First of all we know that $\lim_{n \to \infty} g_n(x) = 0 = g(x)$ over the entire interval $x \in [0,1]$.

If we choose ϵ, then for convergence we need $|g(x) - g_n(x)| < \epsilon$ for all $n > N$. If N depends only on ϵ, then the convergence is uniform, whereas if it depends on both ϵ and x then it is pointwise.

Then we find: $|g(x) - g_n(x)| = \left|0 - \frac{x^n}{n}\right| = \frac{x^n}{n} < \epsilon$

Now we can clearly see that $\frac{x^n}{n} \leq \frac{1}{n}$ for all $x \in [0,1]$. So we can argue that as long as $\frac{1}{n} < \epsilon$, or $n > \frac{1}{\epsilon}$, then certainly $\frac{x^n}{n} < \epsilon$. So the sequence converges uniformly, and hence it also converges pointwise and in the mean.