Welcome!!

This is Mathematical Methods in Physics. What type of course is this? Well I looked up a few examples, and they all looked radically different. My guess is that faculty adapt it to methods they use or are comfortable with. Well I would love to include algebraic and differential topology as well as K-theory, but no. I am gonna focus on what is most useful in general.

Math starts w/ "definitions" (indicated by [ ) which require no proof. Based on these starting points there are then interesting consequences or "theorems" (indicated by [ ) which must be proven using only the definitions or other previously proven theorems. Green means go ahead, Red means stop and prove me.

Often a theorem uses "if and only if" or "iff". The "if" means necessary, while the "only if" means sufficient. Let's see the difference between these:

1. A if B : $A \Leftarrow B$
2. A only if B : $A \Rightarrow B$
3. A if and only if B : $A \Leftrightarrow B$

What are vectors?

- Things with multiple components
- Things with direction and magnitude
- Things you can multiply to get scalars
- Things you can multiply to get vectors
- Column matrices
- Things you can rotate in space
- Things that can be expended in an orthonormal basis
- Things you can add/subtract
- Things you can multiply with a scalar

Let's get a formal definition:

e.g.s $\mathbb{R}$ w/ $+$ $e=0$

$\mathbb{R}-0$ w/ $\times$ $e=1$

$\underbrace{M}_{\text{subject to conditions}}$ w/ MX $e=I$

A group is a system $\{G, \bullet\}$ that consists of a set G w/ a single operation $\bullet$ that satisfies:

1. $\bullet$ is closed, i.e. for $a, b \in G$, $a \bullet b = c \in G$
2. $\bullet$ is associative, i.e. for $a, b, c \in G$, $a \bullet (b \bullet c) = (a \bullet b) \bullet c$
3. There exists an identity $e \in G$ s.t. for all $a \in G$, $a \bullet e = e \bullet a = a$
4. For every $a \in G$ there exists $a^{-1} \in G$ s.t. $a \bullet a^{-1} = a^{-1} \bullet a = e$

Note that we didn't require $a \bullet b = b \bullet a$, if we did then the group is abelian.

Why do we need a group? Well we'll get there... but first

e.g.s $\mathbb{R}$ w/ $+$ and $\times$

$\mathbb{C}$ w/ $+$ and $\times$

A field is a system $\{F, +, \bullet\}$ that satisfies:

1. The subsystem $\{F, +\}$ is an abelian group w/ $e = 0$
2. Let F' be all $x \in F$ except $x = 0$. Then the subsystem $\{F', \bullet\}$ is an abelian group w/ e'.
3. For $a, b, c \in F$, $a \bullet (b + c) = a \bullet b + a \bullet c$, i.e. $\bullet$ is distributive w.r.t. $+$

Now we're ready!

A <u>vector space</u> over a field F is the set of vectors V satisfying:

1. $\{V, +\}$ forms an abelian group w/ $e = 0$
2. For every $\alpha \in F$ and $x \in V$ there exists an element $\alpha x \in V$ and
   a) $\alpha(\beta x) = (\alpha\beta)x \quad \alpha, \beta \in F, x \in V$
   b) $\alpha(x + y) = \alpha x + \alpha y \quad \alpha \in F, x, y \in V$
   c) $1(x) = x$ for all $x \in V$
   d) $(\alpha + \beta)x = \alpha x + \beta x \quad \alpha, \beta \in F, x \in V$

Examples:

1. An n-tuple of real numbers, $x = (\beta_1, \beta_2, \cdots, \beta_n)$ over $F = \mathbb{R}$
   w/ $x + y = (\beta_1, \beta_2, \cdots, \beta_n) + (\delta_1, \delta_2, \cdots, \delta_n) = (\beta_1 + \delta_1, \beta_2 + \delta_2, \cdots, \beta_n + \delta_n)$
   and $\alpha x = \alpha(\beta_1, \beta_2, \cdots, \beta_n) = (\alpha\beta_1, \alpha\beta_2, \cdots, \alpha\beta_n)$
   $e = 0 = (0, 0, \cdots, 0)$. We'll call this $F^n = \mathbb{R}^n$.
2. For $n = 1$ in the above, the vector space is $V = \mathbb{R}$ over $F = \mathbb{R}$ w/ $+$ and $\times$.
3. An n-tuple of complex numbers, $x = (\beta_1, \beta_2, \cdots, \beta_3)$ over $F = \mathbb{C}$ w/ same conditions as in (1). We'll call this $F^n = \mathbb{C}^n$.
4. For $n = 1$ in (3), the vector space is $V = \mathbb{C}$ over $F = \mathbb{C}$ w/ $+$ and $\times$.
5. The set of all polynomials to order n that are functions of a real variable t with real coefficients, so we have $V = P_n$ over $F = \mathbb{R}$.
6. The set of all polynomials to order n that are functions of a real variable t with complex coefficients, so we have $V = P_n$ over $F = \mathbb{C}$.

Let's consider (5) and (6). We have for $x \in P_n$, $x = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_n t^n$ where $\alpha \in \mathbb{R}$ for (5) and $\alpha \in \mathbb{C}$ for (6).

Start w/ $\{P_n, +\}$ which should form an abelian group. Let's check:

For $x, y, z \in P_n$
$$x + y = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_n t^n + \beta_0 + \beta_1 t + \beta_2 t^2 + \cdots + \beta_n t^n$$
$$= (\alpha_0 + \beta_0) + (\alpha_1 + \beta_1)t + (\alpha_2 + \beta_2)t^2 + \cdots + (\alpha_n + \beta_n)t^n \in P_n \quad \underline{\text{closed}}$$
$$(x + y) + z = (\alpha_0 + \beta_0) + \gamma_0 + \cdots = \alpha_0 + (\beta_0 + \gamma_0) + \cdots = x + (y + z) \quad \underline{\text{associative}}$$
$$e = 0 \text{ so for any } x \in P_n \quad 0 + x = x + 0 = x \quad \underline{\text{identity}}$$
$$x^{-1} = -x = -\alpha_0 - \alpha_1 t - \alpha_2 t^2 - \cdots - \alpha_n t^n \text{ s.t. } x^{-1} + x = x + x^{-1} = 0 = e \quad \underline{\text{inverses}}$$
$$x + y = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_n t^n + \beta_0 + \beta_1 t + \beta_2 t^2 + \cdots + \beta_n t^n$$
$$= \beta_0 + \beta_1 t + \beta_2 t^2 + \cdots + \beta_n t^n + \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_n t^n = y + x \quad \underline{\text{abelian}}$$

Furthermore let's consider $x \in P_n$ and $\gamma \in \mathbb{R}$ as in (5), then clearly
$\gamma x = \gamma\alpha_0 + \gamma\alpha_1 t + \gamma\alpha_2 t^2 + \dots + \gamma\alpha_n t^n \in P_n$ (similarly for (6)),

moreover:

a) $\gamma(\delta x) = \gamma(\delta\alpha_0 + \delta\alpha_1 t + \dots + \delta\alpha_n t^n) = \gamma\delta\alpha_0 + \gamma\delta\alpha_1 t + \dots + \gamma\delta\alpha_n t^n$
$= \gamma\delta(\alpha_0 + \alpha_1 t + \dots + \alpha_n t^n) = (\gamma\delta)x$ ✓

b) $1(x) = 1(\alpha_0 + \alpha_1 t + \dots + \alpha_n t^n) = \alpha_0 + \alpha_1 t + \dots + \alpha_n t_n = x$ ✓

c) $\gamma(x+y) = \gamma(\alpha_0 + \alpha_1 t + \dots + \alpha_n t^n + \beta_0 + \beta_1 t + \dots + \beta_n t^n)$
$= \gamma\alpha_0 + \gamma\alpha_1 t + \dots + \gamma\alpha_n t^n + \gamma\beta_0 + \gamma\beta_1 t + \dots + \gamma\beta_n t^n = \gamma x + \gamma y$ ✓

d) $(\gamma+\delta)x = (\gamma+\delta)(\alpha_0 + \alpha_1 t + \dots + \alpha_n t^n) = \gamma\alpha_0 + \gamma\alpha_1 t + \dots + \gamma\alpha_n t^n + \delta\alpha_0 + \delta\alpha_1 t + \dots + \delta\alpha_n t^n$
$= \gamma x + \delta x$ ✓

Back to what we said about vectors:

nope ≡ not required

- Things with multiple components — nope
- Things with direction and magnitude — nope
- Things you can multiply to get scalars — nope
- Things you can multiply to get vectors — nope
- Column matrices — nope
- Things you can rotate in space — nope
- Things that can be expanded in an orthonormal basis — nope
- Things you can add/subtract — yes
- Things you can multiply with a scalar — yes

To organize our handling of vectors, it is convenient to have a basis.
Should it be orthonormal? Well we don't even have an inner product yet.
Instead:

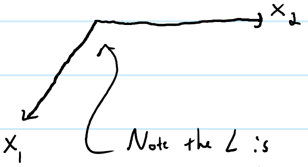
A <u>basis</u> in a vector space V is a set $\{x_i\} \in V$ of <u>linearly independent</u> vectors that <u>spans</u> the space, i.e. any element of V is a linear combination of $\{x_i\}$

Linear independence relies only on vector addition, and no product. Let's see...

A set of n vectors $\{x_i\}$ is linearly independent if and only if $\sum_{i=1}^{n} \alpha_i x_i = 0 \Rightarrow \underbrace{\alpha_i = 0}_{\text{for all } i}$

Note that we are not requiring "spanning" just yet. Let's look at some examples.

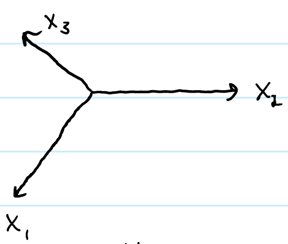
In $\mathbb{R}^3$



Note the $\angle$ is irrelevant as long as it is not $0, 180^\circ$

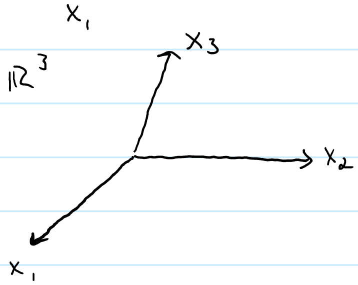
$\Rightarrow \sum_{i=1}^{2} \alpha_i x_i = 0 \rightarrow \alpha_i = 0$

In $\mathbb{R}^3$



All three are coplanar $\Rightarrow \sum_{i=1}^{3} \alpha_i x_i = 0$ w/ $\alpha_i \neq 0$

In $\mathbb{R}^3$



The three are not coplanar $\Rightarrow \sum_{i=1}^{3} \alpha_i x_i = 0 \rightarrow \alpha_i = 0$

Note this is 4D

For P_3 consider $x_1(t) = t - t^3$, $x_2(t) = 2 + t^2$, $x_3(t) = -1 + t$, $x_4(t) = -2 + 2t^3$.
Linearly independent or not? You decide.
Answer: $\alpha_1 = -2, \alpha_2 = 0, \alpha_3 = 2, \alpha_4 = -1 \Rightarrow \sum \alpha_i x_i = 0$ so no.

Of course the criterion for a linearly independent set of vectors can be put in a more familiar fashion:

The set of nonzero vectors $\{x_i\}$ is linearly dependent if and only if some x_i, $2 \leq i \leq n$ is a linear combination of the preceding ones.

Let's see a proof (even though the result should be intuitive).

if (necessary): Suppose that if we start with only one of the vectors. This is obviously a linearly independent set since $\alpha_1 x_1 = 0 \Rightarrow \alpha_1 = 0$. Now consider more than one. Clearly a subset can be linearly ind. (at least a subset of 1). There must be a smallest number (k) for which we transition from a lin. ind. set to a lin. dep. set. That is, up to $k-1$, $\sum_{i=1}^{k-1} \alpha_i x_i = 0 \Rightarrow \alpha_i = 0$, but $\sum_{i=1}^{k} \alpha_i x_i = 0 \Rightarrow \alpha_i \neq 0$. But this means that α_k itself $\neq 0$, otherwise we would have $\underbrace{\sum_{i=1}^{k-1} \alpha_i x_i = 0 \Rightarrow \alpha_i \neq 0}_{\text{which violates it being lin. ind.}}$. If $\alpha_k \neq 0$, then $x_k = \underbrace{-\frac{1}{\alpha_k} \sum_{i=1}^{k-1} \alpha_i x_i}_{\text{as promised}}$

only if (sufficient): If we start w/ $x_k = \sum_{i=1}^{k-1} \alpha_i x_i$, then we just set $\alpha_k = -1$ then we have $\sum_{i=1}^{k} \alpha_i x_i = \sum_{i=1}^{k-1} \alpha_i x_i + \alpha_k x_k = \sum_{i=1}^{k-1} \alpha_i x_i - x_k = 0$

Then if we set $\alpha_{k+1} = \alpha_{k+2} = \cdots = \alpha_n = 0$ we have $\underbrace{\sum_{i=1}^{n} \alpha_i x_i = 0 \text{ w/ some } \alpha_i \neq 0}_{\text{the condition of lin. dep.}}$.

Back to spanning. A basis requires a set of linearly independent vectors which span the space.

So for a few examples:

In $\mathbb{R}^3$ clearly $(1,0,0), (0,1,0), (0,0,1)$ forms a basis, but so does $(1,0,0), (0,1,0), (1,1,1)$.

In P_3 clearly $1, t, t^2, t^3$ forms a basis, but so does $1, t, t^2, 1+t+t^2+t^3$.

Some more useful consequences of the "basis" definition.

[Every vector has a unique representation as a linear combination of a fixed basis.

Proof: Suppose there were 2, then $x = \sum_i \alpha_i x_i$ and $x = \sum_i \beta_i x_i$, then $x - x = \sum_i \alpha_i x_i - \sum_i \beta_i x_i = \sum_i (\alpha_i - \beta_i) x_i = 0$, but recall that the only way to have $\Sigma = 0$ w/ a linearly independent set is that all of the coefficients $= 0$, hence $\alpha_i = \beta_i$. Q.E.D.

Given that the number of elements in any basis of a given vector space is the same (which can and is proven in the text), then this leads us to the useful definition:

[The dimension of a vector space is the number of elements in any basis.

A quick side-note: Since we may always consider a vector space as the flowering of pointy objects from the origin, we may interchange the notion of a vector w/ the location of its point in the space. Therefore $\underbrace{\text{components}}_{\text{as used by me}} \leftrightarrow \underbrace{\text{coordinates}}_{\text{as used by author}}$.