At this point we can go in two directions:

- Keep things finite and explore space(time) vector structures
- Let $n \to \infty$ and explore consequences

We'll do both, but let's start with $n \to \infty$.

Imagine the space of polynomials in t up to order n, i.e. $x(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \dots + \alpha_n t^n$

Consider the vector $x(t) = \sum_{i=0}^{n} \frac{2^i}{i!} t^i$

If we allow $n=1$ this is simple, so is $n=2$, $n=3$.
But consider $n = 6,927,436,221$. Bigger sucks worse right?
Well what about $n \to \infty$? $x(t) = \sum_{i=0}^{\infty} \frac{2^i}{i!} t^i = e^{2t}$

Reversing this logic, perhaps we can express any function as a vector in a vector space. And perhaps these vector spaces have better bases than the nonorthonormal (but linearly independent) set $\{1, t, t^2, \dots, t^\infty\}$.
If so, then we can employ the power of vector spaces, linear transformations, etc.

Aiming to generalize our analysis of finite vector spaces will lead us to the idea of Hilbert space, and all the special functions that live there-on.

Getting started we first need two things:

[A closed interval $[a,b]$ is the set of all points $a \le x \le b$. Open (a,b) would mean $a < x < b$.

[A function is square integrable (s.i.) on $[a,b]$ if $\int_a^b \underbrace{|f(x)|^2}_{f^*(x)f(x)} dx$ exists and is finite.

Now:

[We will begin w/ function space, that is the space of complex-valued functions of a real variable x, defined on a closed interval $[a,b]$ which are square integrable. It is a vector space.

Recall:

[A <u>vector space</u> over a field F is the set of vectors V satisfying:

1. $\{V, +\}$ forms an abelian group w/ $e = 0$
2. For every $\alpha \in F$ and $x \in V$ there exists an element $\alpha x \in V$ and
   - a) $\alpha(\beta x) = (\alpha\beta)x \quad \alpha, \beta \in F,\ x \in V$
   - c) $1(x) = x$ for all $x \in V$
   - b) $\alpha(x+y) = \alpha x + \alpha y \quad \alpha \in F,\ x, y \in V$
   - d) $(\alpha + \beta)x = \alpha x + \beta x \quad \alpha, \beta \in F,\ x \in V$

In function space, the addition of two vectors is defined as: $(f_1 + f_2)(x) \equiv f_1(x) + f_2(x)$

and complex scaler multiplication of a vector is defined by: $(\alpha f)(x) \equiv \alpha f(x)$ Not $(\alpha f)x = f(\alpha x)$!!

If $f(x) = x^2 \Rightarrow (\alpha f)x = \alpha x^2$ ☺ If $f(x) = x^2 \Rightarrow (\alpha f)x = \alpha^2 f(x)$ ☹

These are the two basic operations that define a vector space. The only worry is closure, i.e. that the sum of two vectors on the space is another vector on the space, in this case meaning is s.i. + s.i. = s.i.?

Well consider:

$$|f_1 + f_2|^2 = (f_1^* + f_2^*)(f_1 + f_2)$$

$$= |f_1|^2 + |f_2|^2 + f_1^* f_2 + f_1 f_2^*$$

$$= |f_1|^2 + |f_2|^2 + 2\,\mathrm{Re}(f_1^* f_2)$$

$$\le |f_1|^2 + |f_2|^2 + 2\underbrace{|f_1^* f_2|}_{\sqrt{[\mathrm{Re}(f_1^* f_2)]^2 + [\mathrm{Im}(f_1^* f_2)]^2}} = |f_1|^2 + |f_2|^2 + 2\underbrace{|f_1||f_2|}_{\sqrt{f_1^* f_1}\sqrt{f_2^* f_2}}$$

$$(a - ib)(\overbrace{c+id}^{f_2}) + (\overbrace{a+ib}^{f_1})(c-id) = ac + bd - ibc + iad + ac + bd + ibc - iad = 2ac + 2bd = 2\,\mathrm{Re}(f_1^* f_2)$$

if $f_1 = a + ib$, $f_2 = c + id$: $\sqrt{(ac+bd)^2 + (ad-bc)^2}$; $\sqrt{a^2+b^2}\sqrt{c^2+d^2}$

$$\sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 + \cancel{acbd} - \cancel{acbd}} = \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}$$

but $(|f_1| - |f_2|)^2 = |f_1|^2 + |f_2|^2 - 2|f_1||f_2| \ge 0 \Rightarrow |f_1|^2 + |f_2|^2 \ge 2|f_1||f_2|$

so $|f_1 + f_2|^2 \le 2|f_1|^2 + 2|f_2|^2 \Rightarrow \underbrace{\int_a^b |f_1 + f_2|^2 dx}_{\text{finite}} \le 2\underbrace{\int_a^b |f_1|^2 dx}_{\text{finite}} + 2\underbrace{\int_a^b |f_2|^2 dx}_{\text{finite}}$ ⇐

Now part of the above includes products of functions, e.g. $f_1^* f_2$. These should be understood as $f_1^*(x) f_2(x) = g(x)$ instead of $f_1^*(f_2(x))$.

Now let's go ahead and define an inner product:

An inner-product in a real or complex vector space is a scalar valued function of the ordered pair of vectors x and y s.t.

1. $(x, y) = (y, x)^*$ [If they are real then order doesn't matter]
2. $(\alpha x + \beta y, z) = \alpha^*(x, z) + \beta^*(y, z)$ w/ α, β scalars
3. $(x, x) \geq 0$ for any x; $(x, x) = 0 \Rightarrow x = 0$ "positive-definite, which can be relaxed"

We can use:

$$(f_1, f_2) \equiv \int_a^b f_1^*(x) f_2(x) dx$$

Note that s.i. basically uses this for a function w/ itself: $(f, f) = \int_a^b |f|^2 dx < \infty$

We call $\|f\| = \sqrt{(f, f)}$ the "norm" of f.

Now the inner product of two different s.i. functions always exists and is finite: because f_1, f_2 are s.i.

Recall: $|f_1^* f_2| = |f_1||f_2| \leq \frac{1}{2}(|f_1|^2 + |f_2|^2) \Rightarrow \int_a^b |f_1^* f_2| dx \leq \frac{1}{2}(\|f_1\|^2 + \|f_2\|^2) < \infty$

but $\int_a^b f_1^* f_2 dx \leq \int_a^b |f_1^* f_2| dx < \infty$

The inner product defined above clearly follows all the rules in defining an inner product, except for positive-definiteness.

Clearly $(f, f) \geq 0$, but does $(f, f) = 0 \Rightarrow f = 0$?

More precisely, does $(f, f) = 0 \Rightarrow f(x) = 0$ for all x? Turns out that to ensure this requires a redefinition of an "integral"!

Dirichlet function

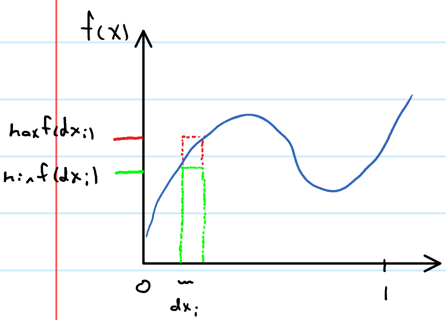
Consider the function $D(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$ over the interval $x \in [0,1]$.

What is $(\sqrt{D},\sqrt{D})$ w/ $a=0$, $b=1$ equal to? First and foremost, it seems that to satisfy the condition above, we need $(\sqrt{D},\sqrt{D}) \neq 0$ since $D(x) \neq 0$ for some x.

Now $(\sqrt{D},\sqrt{D}) = \int_0^1 \sqrt{D}^* \sqrt{D}\, dx = \int_0^1 D\, dx$ since $D(x)$ is real.

But how do we integrate it? Let's try a good old set of Riemann rectangles.

Recall:

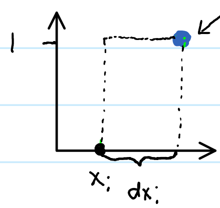


$$\int_0^1 f(x) = \lim_{dx_i \to 0} \sum_i dx_i\, f(x_i) \quad \begin{array}{l} \nearrow \max f(x_i) \\ \searrow \min f(x_i) \end{array}$$

Normally, it doesn't matter which we take, max or min, and in fact the integral is well-defined if taking either yields the same result.

Now for the function $D(x)$ let's suppose we use the rectangle method.

In this case:



the interval includes a rational # and as $dx_i \to 0$ it will never include more than one!

$$\Rightarrow \int_0^1 D(x)\,dx = \lim_{dx_i \to 0} \sum_i \begin{cases} dx_i\, 1 & \max f(x_i) \\ dx_i\, 0 & \min f(x_i) \end{cases}$$

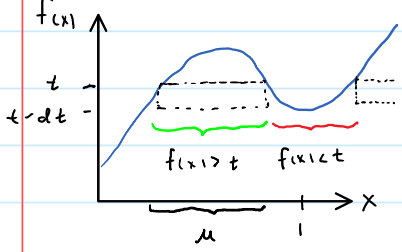
Clearly for $\min f(x_i) \Rightarrow \int_0^1 D(x)\,dx = 0$

For $\max f(x_i) \Rightarrow \int_0^1 D(x)\,dx = 1$

This means the Riemann integral for the Dirichlet function does not exist!

Can save it? Yes we can, by redefining the integral into one which can handle all the functions that Riemann can, and more. This is the Lebesgue integral.

Instead of rectangles over dx_i, Lebesgue instead breaks the area under a curve into horizontal strips.



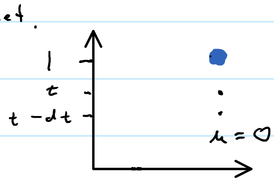
To get the area of each strip we start w/ their height which we will parameterize generically by t, so the height is dt. For the width we use the following: $\mu(x, f(x)>t)$ which gives us the lengths along x for regions where the curve is higher than t.

Now if you think about it, $\int_0^\infty \mu(x, f(x)>t)_{ab}\, dt = \int_a^b f(x)\,dx$ where μ_{ab} only does its job over the range of x from a to b.

Now μ defines a measure of integration, and so we can also write $\int_a^b f(x)\,dx = \int_{ab} f\,d\mu$.

Here's an important observation: Anytime $\int_a^b f(x)\,dx$ exists, so to does $\int_{ab} f\,d\mu$, that is wherever the Riemann exists, so to does Lebesgue, and they give the same answer.

Even more important: Lebesgue's exist even when Riemann's don't. As in the case of Dirichlet.



That is, the integral $= 0$ since the measure of integration $d\mu$ always $= 0$. This is called 0 "almost everywhere" and is handled as the 0 function.

Yet another: Lebesgue integrals nicely commute w/ limits, i.e. $\lim_{u\to 0} \int f(t,u)\,dt = \int \lim_{u\to 0} f(t,u)\,dt$

So $\langle f, f\rangle = 0 \Rightarrow f = 0$ almost everywhere, and our inner-product is well-defined.

For Dirichlet, $\langle \sqrt{D}, \sqrt{D}\rangle = 0 \Rightarrow D = 0$ almost everywhere.

We will not need the Lebesgue integral in practice, as the functions we encounter will be Riemann integrable. But we use Lebesgue to define the function space.

Now we need the space to be complete. What does that mean? We should start by saying that completeness here is a statement about the function space, and not so much its vector realization (though we will come to that).

Completeness of a space is the following:

A complete space is one in which there exists no Cauchy sequence of elements of the space which tends towards limits outside of the space.

A Cauchy sequence $\{s_n\}$ is such that given a number $\epsilon > 0$, then there is some index $N(\epsilon)$ s.t. if n and m are larger than $N(\epsilon)$, then $\|s_n - s_m\| < \epsilon$.

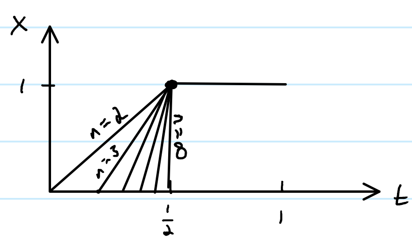
An example of an incomplete space. Consider X to be the space of continuous functions on $[0,1]$ w/ norm defined by $\|x\| = \int_0^1 |x(t)|\,dt$ (note this is different than our space).

Define a sequence of elements of X:

$$x_n(t) = \begin{cases} 0 & \text{for } 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ nt - \frac{n}{2} + 1 & \text{for } \frac{1}{2} - \frac{1}{n} \le t \le \frac{1}{2} \\ 1 & \text{for } t \geq \frac{1}{2} \end{cases}$$

This is Cauchy since: $\|x_n - x_m\| = \frac{1}{2}\left|\frac{1}{n} - \frac{1}{m}\right| < \epsilon$ if for $m \to \infty$ (which gives the largest value) $\frac{1}{2n} < \epsilon \Rightarrow n > \frac{1}{2\epsilon} \cong N(\epsilon)$

But consider these drawn:



But is $n \to \infty$ continuous? Nope!

$$x_{n\to\infty}(t) = \begin{cases} 0 & \text{for } 0 \le t < \frac{1}{2} \\ 1 & \text{for } t \geq \frac{1}{2} \end{cases}$$

So the worry is whether our definition of a space of <u>square integrable</u> functions is complete. That is, is there any Cauchy sequence of s.i. functions whose limit is not s.i.?

According to Riesz and Fischer, the answer is no, so our space is complete.

Let the functions $f_1(x), f_2(x), \dots$ be a sequence.

If $\lim_{n,m\to\infty} \|f_n - f_m\|^2 \equiv \lim_{n,m\to\infty} \int_a^b |f_n(x) - f_m(x)|^2 dx = 0$, then there exists a square (Lebesgue) integrable function $f(x)$ to which the sequence $f_n(x)$ converges "in the mean."

That is, $\lim_{n\to\infty} \int_a^b |f(x) - f_n(x)|^2 dx = 0$ (the proof is h.a.f.)

So the space of s.i. functions w/ our inner-product is complete. It's called Hilbert space.

We define orthogonality, normalization and orthonormality w/ the inner product as usual:

$(f_i, f_j) \equiv \int_a^b f_i^*(x) f_j(x) dx = \delta_{ij}$ $\Rightarrow$ $\{f_i\}$ is an orthonormal set

Recall we can generalize this (and soon will) to use a weighted inner-product:

$$\langle f_i, f_j \rangle_w \equiv \int_a^b f_n^*(x) f_m(x) \underbrace{w(x)}_{\text{same function for any } f_i \text{ and } f_j} dx = \delta_{ij}$$

As an example of an orthonormal set (though not necessarily a complete basis) consider the Fourier functions:

$f_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ $n = 0, \pm 1, \pm 2, \cdots$ over the interval from $-\pi$ to π

Then:

$$\langle f_m, f_n \rangle = \int_{-\pi}^{\pi} f_m^* f_n dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)x} dx = \frac{1}{2\pi i(n-m)} e^{i(n-m)x} \Big|_{-\pi}^{\pi} = 0 \text{ if } n \neq m$$

$$\langle f_n, f_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^0 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx = 1$$