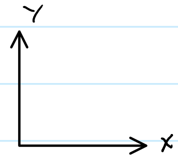
Complex Numbers vs. $\mathbb{R}^2$



$\vec{r} = (x, y) \in \mathbb{R}^2$

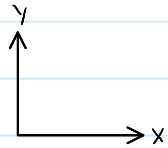
$\vec{r}_1 + \vec{r}_2 = (x_1 + x_2, y_1 + y_2)$

$k\vec{r} = (kx, ky)$ k-real

$\vec{r}_1 \cdot \vec{r}_2 = x_1 x_2 + y_1 y_2 \in \mathbb{R}$

$\vec{r}_1 \times \vec{r}_2 = ?$

The difference



$z = x + iy = (x, y) \in \mathbb{C}$

$z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$

$kz = (kx, ky)$ k-real

$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) \in \mathbb{C}$

Analytic Functions

Lets start w/ an arbitrary complex-valued function of a complex variable:

$w(z) = u(x, y) + i v(x, y)$ w/ $z = x + iy$

Now while $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ obviously make sense and are well defined, what we are really interested in is $\frac{dw}{dz}$. This carries more restrictions.

[$w(z)$ is continuous at z_0 if for $\epsilon > 0$ there exists a δ s.t. $|w(z) - w(z_0)| < \epsilon$ for $|z - z_0| < \delta$

or replace z w/ $z_0 + \Delta z$

and

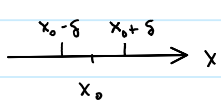
[$w(z)$ is differentiable at z_0 if the limit $\lim_{z \to z_0} \frac{w(z) - w(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = w'(z_0)$ exists.

or better still

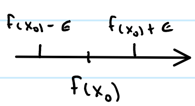
[$w(z)$ is differentiable at z_0 if for $\epsilon > 0$ there exists a δ s.t. $\left| w'(z) - \frac{w(z) - w(z_0)}{z - z_0} \right| < \epsilon$ for $|z - z_0| < \delta$

The primary difference between these and similar definitions for $f(x)$, $f'(x)$ $x \in \mathbb{R}$ are the paths of approach to x_0 and z_0.

Clearly for $f(x)$ we approach x_0:

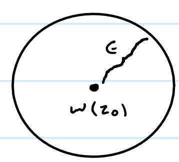


$\Rightarrow$



Whereas for $w(z)$ we approach z_0:

$\Rightarrow$



It is these "isotropic" derivatives which make analytic functions so special.

A single valued function $w(z_0)$ is analytic (or regular) at z_0 if the derivative at z_0 and in a small neighborhood around it exists. If analytic over all of $\mathbb{C}$, then $w(z)$ is "entire."

Consider:

1. $w(z) = z^* = x - iy \Rightarrow$ in terms of Δz, $w'(z_0) = \lim_{\Delta z \to 0} \frac{w(z_0+\Delta z) - w(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{z_0^* + \Delta z^* - z_0^*}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z^*}{\Delta z}$

$$\left.\begin{array}{l} \text{If } \Delta z \to 0 \text{ along } x \Rightarrow \Delta z = \Delta x,\ \Delta z^* = \Delta x \Rightarrow w'(z_0) = 1 \\ \text{If } \Delta z \to 0 \text{ along } y \Rightarrow \Delta z = i\Delta y,\ \Delta z^* = -i\Delta y \Rightarrow w'(z_0) = -1 \end{array}\right\} \text{Not differentiable} \Rightarrow \text{Not analytic}$$

2. $w(z) = z^2 = x^2 - y^2 + i2xy \Rightarrow w'(z_0) = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} = \lim_{\Delta z \to 0} (2z_0 + \Delta z) = 2z_0$

Which clearly does not depend on the "path of approach" at all $\Rightarrow w(z)$ is analytic

3. $w(z) = |z|^2 = x^2 + y^2 \Rightarrow w'(z_0) = \lim_{\Delta z \to 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z} = \lim_{\Delta z \to 0} \overbrace{\Delta z^*}^{\to 0} + \overbrace{z_0^*}^{=0} + \overbrace{z_0}^{=0} \frac{\Delta z^*}{\Delta z} = \begin{cases} 0 & \text{for } z_0 = 0 \\ \text{undetermined} & \text{for } z_0 \neq 0 \end{cases}$ (for $z_0 = 0$)

So this one is differentiable at $z_0 = 0$, but not analytic.

It may seem hard to find (or identify) analytic functions. Especially when written in terms of (x,y). Well, we do have a list of useful results, and then a simpler test to use.

Results:

Everything below is analytic

1. $w(z) = kz^n \quad n = 0, 1, 2, \ldots \quad k \in \mathbb{R}$
2. Sum, product and quotient of 2 analytic functions (provided denominator $\neq 0$)
3. $f(w(z))$ if $w(z)$ is analytic and $f(z)$ is analytic
4. w if $\frac{\partial w}{\partial z^*} = 0$, i.e. $w(z)$ is only a function of z, not z^* (see 1 + 3 above)

Simpler test:

$w(z) = u(x,y) + iv(x,y)$ w/ $z = x + iy$

$$w'(z_0) = \lim_{\Delta z \to 0} \left(\frac{\Delta u}{\Delta z} + i \frac{\Delta v}{\Delta z} \right) \begin{cases} \xrightarrow{\Delta y = 0,\ \Delta z = \Delta x} \dfrac{\partial u}{\partial x} + i \dfrac{\partial v}{\partial x} \\ \qquad\qquad \parallel \\ \xrightarrow{\Delta x = 0,\ \Delta z = i\Delta y} \dfrac{\partial v}{\partial y} - i \dfrac{\partial u}{\partial y} \end{cases} \Bigg\} \ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{Cauchy-Riemann}$$

Thus for differentiability of $w(z)$ at z_0, CR and first partials of u and v exist and are continuous.

Now it turns out that satisfying CR makes things quite pretty.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 u}{\partial y^2} \Rightarrow \overbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \underbrace{\nabla^2 u}_{\text{called harmonic}} = 0}^{\text{Laplace eqn.}}, \text{ and } \nabla^2 v = 0 \text{ as well.}$$

If $w = u + iv$ is analytic $\Rightarrow$ u and v are conjugate harmonic functions.

This implies that given one of u or v and if it is harmonic, then we can use CR to find the other up to a constant.

Consider $u(x,y) = 7x^3 - 21xy^2$ which is obviously harmonic. Then we can say

$$\left.\begin{aligned} &\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 21x^2 - 21y^2 = \frac{\partial v}{\partial y} \Rightarrow v(x,y) = \int [21x^2 - 21y^2]\,dy = 21x^2y - 7y^3 + f(x) \\ &\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow 42xy = \frac{\partial v}{\partial x} \Rightarrow v(x,y) = \int [42xy]\,dx = 21x^2y + f(y) \end{aligned}\right\} v(x,y) = 21x^2y - 7y^3$$

Then $w = u + iv = 7x^3 - 21xy^2 + i(21x^2y - 7y^3)$ which is analytic, but $w = z^3$ so that's obvious!

And finally, if there is a point or a curve or a region for which a function is not analytic, we call it singular. The singular places will play an important role shortly.

Now you should be familiar with the basic functions for a real variable x:

x^k, e^x, $\log x$, $\ln x$, $\sin x$, $\cos x$ and all their offspring ($\tan x = \frac{\sin x}{\cos x}$) and properties ($\cos^2 x + \sin^2 x = 1$)

Let's extend these to complex variables z. And since we are just replacing them w/ z, they should be analytic.

1. $w(z) = e^z \equiv e^x(\cos y + i \sin y) \Rightarrow u(x,y) = e^x \cos y \quad v(x,y) = e^x \sin y$

In order to determine $\frac{d}{dz}(e^z)$ we can use the expression from our derivation of CR:

$$w'(z) = \frac{d}{dz}(e^z) = \left.\begin{array}{l} \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y \\ \qquad\qquad\qquad \text{of course} \\ \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = e^x \cos y + ie^x \sin y \end{array}\right\} \frac{d}{dz}(e^z) = e^z \quad \text{(as expected)}$$

We do get the usual $e^{z_1}e^{z_2} = e^{z_1+z_2}$, but moreover we get $e^{z+2\pi i} = e^z e^{2\pi i} = e^z$ so e^z is periodic w/ period $2\pi i$.

2. In trying to define sin(z) and cos(z), imaginary triangles are not gonna help. But we do already know: $\cos y = \frac{1}{2}(e^{iy} + e^{-iy})$ $\xrightarrow{\text{so how bout}}$ $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

$\sin y = \frac{1}{2i}(e^{iy} - e^{-iy})$ $\qquad$ $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

Given that we already know $\frac{d}{dz}(e^z)$ we have:

$\frac{d}{dz}\cos z = \frac{1}{2}(ie^{iz} - ie^{-iz}) = -\frac{1}{2i}(e^{iz} - e^{-iz}) = -\sin z$

$\frac{d}{dz}\sin z = \cos z$

Many things are the same: $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

$\sin(z_1 \pm z_2) = \cos z_1 \sin z_2 \pm \sin z_1 \cos z_2$

$\cos(-z) = \cos z$, $\sin(-z) = -\sin z$

$\cos(z + 2\pi) = \cos z$, $\sin(z + 2\pi) = \sin z$

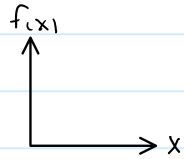
But some are different: $|\sin z|^2 = \sin^2 x + \sinh^2 y \Rightarrow \lim_{y \to \infty} |\sin z|^2 = \infty$

vs.

$|\sin y|^2 \leq 1$

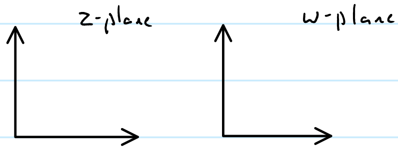
And note that $\tan z = \frac{\sin z}{\cos z}$ is analytic everywhere except when $\cos z = 0$.

Now the good old:



is not going to be useful for $w(z) = u(x,y) + iv(x,y)$

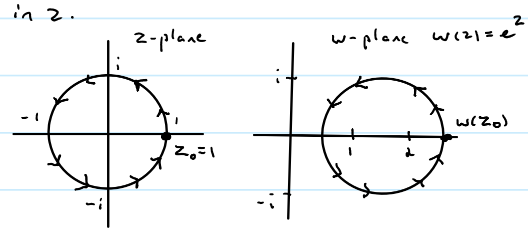
Instead, to visualize things it is helpful to consider:



and instead of inputting the entire z-plane (as we do w/ the x-axis for f(x)), let's just choose a contour of points in z.

For example consider $w(z) = e^z$ and the unit circle in z:

The arrows let us track how w(z) changes w/ z.



Now all the functions thus far have enjoyed that selecting a z_0 and then following a closed contour in z back to z_0 returns the same value of w(z).

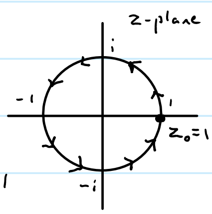
But consider:

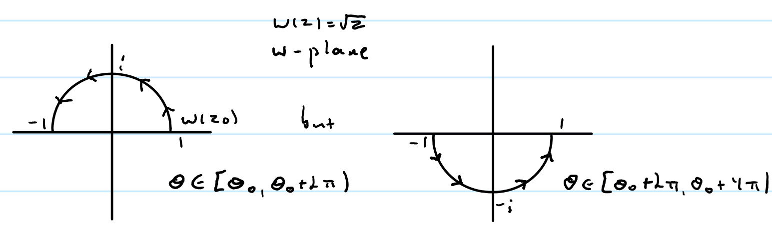
3. $w(z) = \sqrt{z} = \sqrt{x + iy}$ The cleanest way to evaluate this is w/ $z = \underbrace{r\cos\theta}_{x} + i\underbrace{r\sin\theta}_{y} = re^{i\theta}$

then $= \sqrt{r}e^{i\frac{\theta}{2}} = \sqrt{r}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]$

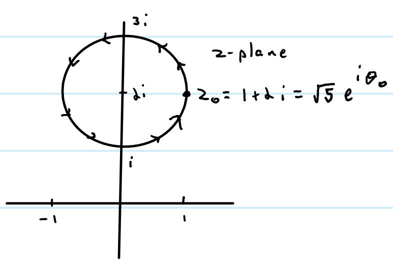
Now consider:

$z = re^{i\theta} = \begin{cases} r = 1 \\ \theta \end{cases}$

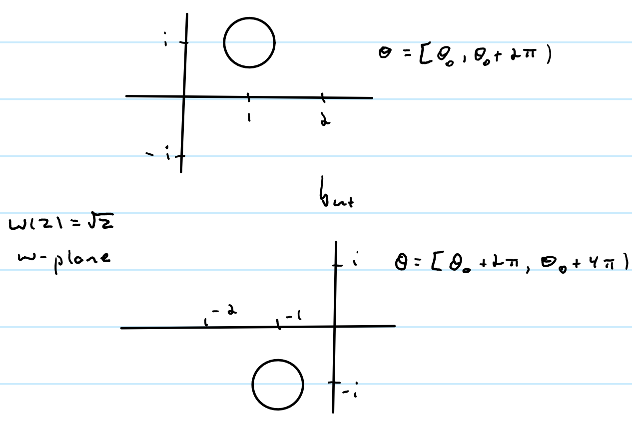




And:



$Z = re^{i\theta} + 2i = \begin{cases} r = 1 \\ \theta \end{cases}$



Clearly something is wrong w/ the origin at $z = 0$ (it is singular), but also the multiplicity of images means that w(z) is multi-valued which is a big no-no for analysis.