Linear Transformations

Okay, so we have vectors. But what can we do to them?

First and foremost we can "transform" them, but not in an arbitrary way (otherwise we might break some of the defining properties of vectors).

A linear transformation/operator A on a vector space V assigns to every vector $x, y \in V$ vectors $Ax, Ay \in V$ s.t.

1. For a, b scalars $A(ax+by) = aAx + bAy$
2. The "product" of two A and B is defined by $ABx \equiv A(\underbrace{Bx}_{\text{a new vector for } A \text{ to act on}})$
3. $(A+B)x \equiv Ax + Bx$

Note that even though the definition of vectors does not include "multiplication" for linear transformations it does!

Let's consider some examples:

1. For F^n (the space of n-tuples), then matrix multiplication MV w/ square $n \times n$ matrices works. We know that $M(ax+by) = aMx + bMy$ for any $x, y \in F^n$ from experience. And obviously $MM'x = M(M'x)$ and $(M+M')x = Mx + M'x$ works as well.

2. Consider P_n (polynomials up to degree n), and the operator $D^k \equiv \frac{d^k}{dt^k}$. Consider $D^k(ax+by) = \frac{d^k}{dt^k}(ax+by) = a\frac{d^k x}{dt^k} + b\frac{d^k y}{dt^k}$ where $\frac{d^k x}{dt^k}, \frac{d^k y}{dt^k} \in P_n$ as well. Also $D^k D^{k'} x = \frac{d^k}{dt^k}\left(\frac{d^{k'}}{dt^{k'}} x\right) = \frac{d^{k+k'}x}{dt^{k+k'}}$ and $(D^k + D^{k'})x = \frac{d^k x}{dt^k} + \frac{d^{k'} x}{dt^{k'}}$.

Why doesn't $Ix = \int x\,dt$ work? Because for P_n, $I t^n = t^{n+1} \notin P_n$.

Two special linear transformations are $Ox = 0$ and $Ix = x$ where the exact form of these depends on the form of the vectors.

The "product" of linear transformations enjoys a host of properties:

a) $AO = OA = O$ c) $A(B+C) = AB + AC$ e) $(aA) = a(A)$ $a \in F$

b) $AI = IA = A$ d) $A(BC) = (AB)C$ Note: $AB = BA$ is not guaranteed!

Inverses

Okay, so for vectors we know that for any $x \in V$, there must exist an $x^{-1} \in V$ s.t. $x + x^{-1} = 0 =$ the identity, i.e. $x + (-x) = 0$.

What about linear transformations? Do they have an inverse? Is it additive or "multiplicative"? (Since L.T.s include addition and "multiplication")

First of all, if we consider a vector space V, then the set <u>of all</u> linear transformations acting on V actually forms a vector space itself!

That is the set $\{A, B, \ldots\} \in V'$ satisfies:

1. There exists an operation $+$ s.t. $\{V', +\}$ forms an abelian group w/ identity $= 0$
2. For every $\alpha \in F$ there exist a transformation $\alpha A \in V'$ and

   a) $\alpha(\beta A) = (\alpha\beta)A$ c) $I(A) = A$ for all $A \in V'$

   b) $\alpha(A + B) = \alpha A + \alpha B$ d) $(\alpha + \beta)A = \alpha A + \beta A$

So yes, <u>there always exists an additive inverse to any linear transformation</u>, i.e. $A + A_+^{-1} = 0$.

What about the "product"? $A A_\bullet^{-1} = I$ does $A_\bullet^{-1}$ exist? First of all let's clean up notation. Since $A_+^{-1} = -A$, we can just call $A_\bullet^{-1} = A^{-1}$.

Here we go...

If a linear transformation A has both the following properties, then A^{-1} exists:

a) $x \neq y \Rightarrow Ax \neq Ay$ (or $Ax = Ay \Rightarrow x = y$)

b) For every $y \in V$ there exists an $x \in V$ s.t. $Ax = y$

Consider the transformation $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ which acts on $\mathbb{R}^2$.

a) $x = \begin{pmatrix} a \\ b \end{pmatrix}$, $y = \begin{pmatrix} c \\ d \end{pmatrix}$ $R_\theta x = \begin{pmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \end{pmatrix}$ and $R_\theta y = \begin{pmatrix} c\cos\theta - d\sin\theta \\ c\sin\theta + d\cos\theta \end{pmatrix}$ we'll use

$$R_\theta x = R_\theta y \Rightarrow \begin{array}{l} a\cos\theta - b\sin\theta = c\cos\theta - d\sin\theta \\ a\sin\theta + b\cos\theta = c\sin\theta + d\cos\theta \end{array} \Rightarrow \begin{array}{l} (a-c)\cos\theta = (b-d)\sin\theta \\ \underline{(a-c)\sin\theta = -(b-d)\cos\theta} \end{array}$$

$\cot\theta = -\tan\theta$ never true!

$\Rightarrow a = c,\ b = d \Rightarrow x = y$

b) $y = \begin{pmatrix} a \\ b \end{pmatrix}$ then $x = \begin{pmatrix} a\cos\theta + b\sin\theta \\ -a\sin\theta + b\cos\theta \end{pmatrix}$ s.t. $R_\theta x = y$ Of course we already know $R_\theta^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

So let's consider a less fortunate example. How about D on P_n?

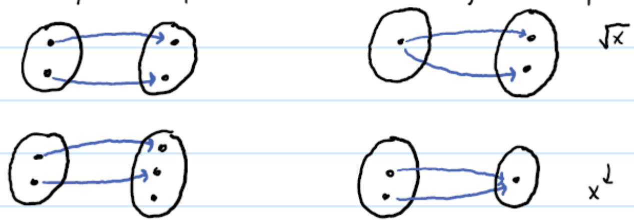
a) $x = t^2 + 2$, $y = t^2 + 3$ $\Rightarrow$ $Dx = 2t = Dy$ but $x \neq y$

b) For $y = t^n + \cdots$ then there exists no x s.t. $Dx = y$

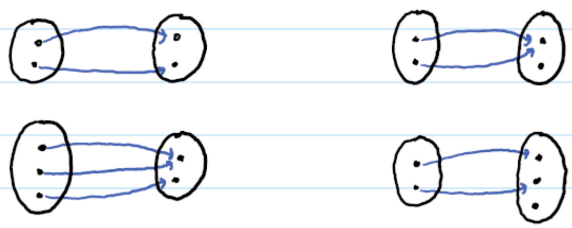
$\Big\}$ So D on P_n has no inverse.

Conditions (a) and (b) correspond to injectivity and surjectivity of the map A.

a) injective (one-to-one) Injective Maps Non-injective Maps

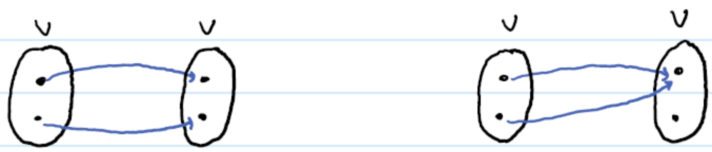


b) surjective (onto) Surjective Maps Non-surjective Maps



Well it turns out that if the two spaces you are mapping between have the same number of elements, then injective $\Leftrightarrow$ surjective! (This is true for finite n)

Surjective and Injective Non-surjective and Non-injective



Just think about why the inverse doesn't exist!

So for a finite dimensional vector space we can choose either condition (a) or (b) to check for an inverse.

So consider the following:

[If $Ax = 0 \Rightarrow x = 0$, then A is invertible.

To show why, just start w/ the first part of the definition, i.e. if $Ax = Ay \Rightarrow x = y$ then A is invertible. Then $Ax - Ay = 0 \underset{\uparrow}{=} A(x-y)$, but if A is invertible this means $x - y = 0$.

using linearity of A

It turns out that if A^{-1} exists, then it satisfies the linearity conditions as well.
Furthermore, there is commutativity between A and A^{-1}, i.e. $AA^{-1} = A^{-1}A = \underline{I}$.

Skip in class

Now hold up, if we consider D on P_n, and introduce $Sx = \int_0^t x(u)\,du$
then for example:

On P_4, $x = t^2 + t \Rightarrow DSx = D\int_0^t (u^2+u)\,du = D\left(\frac{1}{3}t^3 + \frac{1}{2}t^2\right)$

$$= \frac{d}{dt}\left(\frac{1}{3}t^3 + \frac{1}{2}t^2\right) = t^2 + t \quad \text{so} \quad DS = \underline{I}$$

moreover $SDx = S\frac{d}{dt}(t^2+t) = S(2t+1)$

$$= \int_0^t (2u+1)\,du = t^2 + t \quad \text{so} \quad SD = \underline{I}$$

But consider $x = t + 1 \Rightarrow DSx = D\int_0^t (u+1)\,du = D\left(\frac{1}{2}t^2 + t\right)$

$$= \frac{d}{dt}\left(\frac{1}{2}t^2 + t\right) = t + 1 \quad \text{so} \quad DS = \underline{I}$$

but $SDx = S\frac{d}{dt}(t+1) = S(1)$

$$= \int_0^t 1\,du = t \quad \text{so} \quad SD \neq \underline{I}$$

Moreover for $x = t^4 + \cdots$, $Sx \notin V$ since this will be fifth order which is not on P_4.

So again, just as we promised before, S is not a good inverse to D, because D doesn't have one!

To finish up we have:

1. If A and B are invertible, then so is AB w/ $(AB)^{-1} = B^{-1}A^{-1}$.
2. If A is invertible and $\alpha \neq 0$, then $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$.
3. If A is invertible then so is A^{-1} and $(A^{-1})^{-1} = A$.

Note: Please don't take the notation A^{-1} to interpret as division by A.
For numbers it is, i.e. $\alpha^{-1} = \frac{1}{\alpha}$, but not for matrices or other complicated operators.

Isomorphisms

Let's go back to groups for a moment. We can have 2 (or more) groups which are specific examples of a common underlying structure. This means that for each element in group A, there is a corresponding element in group B, and vice versa. Moreover, both sets satisfy the same algebraic structure. If this is the case, these groups are called isomorphic.

To see the algebraic structure of a finite group, we need only its "multiplication" table.

$\{1,-1\}$ w/ $\times$

$\times$	1	−1
1	1	−1
−1	−1	1

$\{E,O\}$ w/ $+$

$+$	E	O
E	E	O
O	O	E

$\{I, R_\pi\}$ w/ $+$

$+$	I	R_π
I	I	R_π
R_π	R_π	I

Note that these all have the same algebraic structure (in fact so does any 2 element group).

But it has to go both ways, so even though we can map rotations in 2D to a subset of rotations in 3D, we cannot map all of the rotations in 3D to rotations in 2D. Therefore rotations in 2D and 3D are not isomorphic.

Now back to vectors. What is interesting about vectors is that they have a well-defined algebraic structure. This will have a consequence in just a moment.

Two vector spaces U and V (over the same field) are isomorphic if there is a 1-to-1 correspondence between $x^{(i)} \in U$ and $y^{(i)} \in V$ (and vice versa) so that we can say $y^{(i)} = f(x^{(i)})$ such that $f(\alpha_1 x^{(1)} + \alpha_2 x^{(2)}) = \alpha_1 f(x^{(1)}) + \alpha_2 f(x^{(2)})$.

But this implies (via proof) something powerful due to the common algebraic structure:

Every n-dimensional vector space V_n over F is isomorphic to F^n.

That is, any n-dimensional vector space over a field F is isomorphic to the vector space composed of n-tuples with their elements coming from F.

An immediate consequence of this is that any two vector spaces w/ the same dimension and over the same field are both isomorphic to F^n and therefore isomorphic to each other.