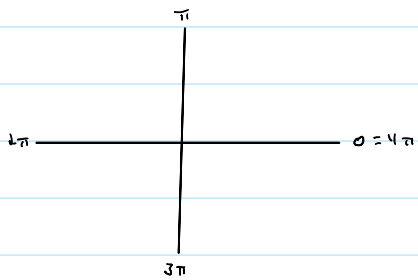
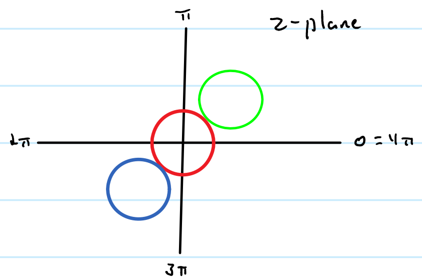
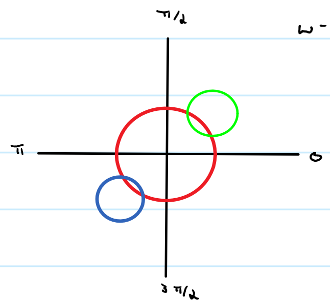
To fix these issues, we can consider extending the $[0, 2\pi)$ "Riemann sheet" by gluing in another.



Note $\perp$ is at $\angle$



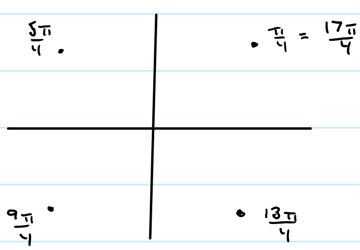


w-plane $w(z) = \sqrt{z} = \sqrt{r}\, e^{i\frac{\theta}{2}}$

Now $w(z)$ is single-valued and analytic everywhere except $z = 0$.

In this case we call $z=0$ the "branch point", and then the positive real axis in z the "branch cut" along which we sever and join in the second copy of $\mathbb{R}^2$.

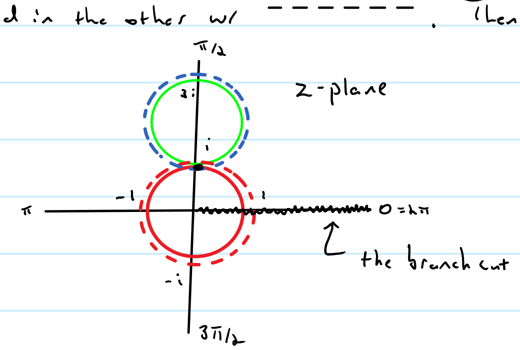
Note, while the branch point is absolute and unchangeable, the branch cut is up to us. We could have done:

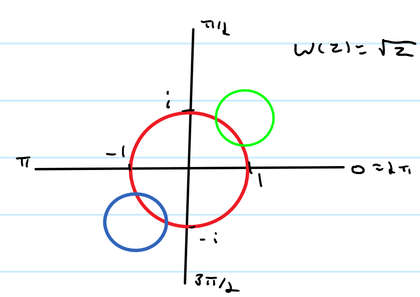


which would have been taking the branch cut along the axis through $1+i$.

Now a slightly better notation for pictoralizing these extended spaces is by denoting paths in one w/ ———— and in the other w/ - - - - - - . Then:

Notice that a contour can either remain entirely within one sheet, or cross over by passing through the cut.

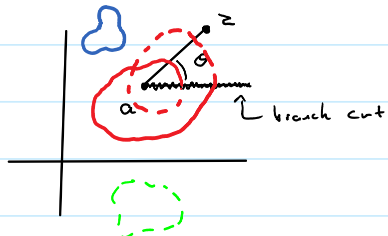




Now the branch point can be modified by choosing a different function.

4. $w(z) = \sqrt{z-a} = |z-a|^{1/2} e^{i\theta/2}$ w/ $z=a$ branch point

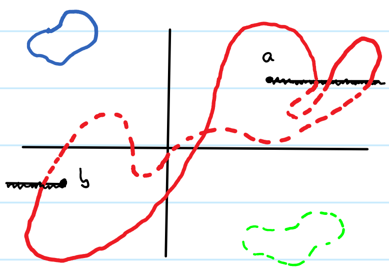
Envisioning this w/ the second method is much easier:



In fact we can also have multiple branch points, each of which can provide us w/ branch cuts into a second surface.

5. $w(z) = \sqrt{(z-a)(z-b)} = |z-a|^{1/2} |z-b|^{1/2} e^{i\theta_1/2} e^{i\theta_2/2}$ w/ $z=a$ and $z=b$ branch points

Then:



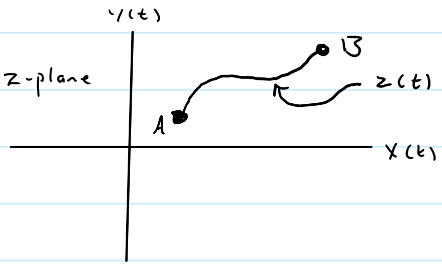
And then of course a function can require more than 2 sheets to be single-valued.

6. $w(z) = \log z \equiv \ln r + i\theta$ w/ $e^{\log z} = e^{\ln r} e^{i\theta} = z$ and $\log(z_1 z_2) = \log z_1 + \log z_2$

Is this periodic? Hell no! The imaginary part grows by 2π everytime we take another trip around $z=0$. How many sheets needed to make it single-valued? ∞ (Draw that!)

The next step in this story is to define integrals.

First of all realize that z spans a plane, unlike x which spans an axis. So to integrate over z, we need to pick a path through the plane.



$z(t)$ s.t. $A = z(t_A)$, $B = z(t_B)$ (or contour C)

Now we divide up the path into small elements and replace each by a straight line. Then we have:

$$\lim_{|\Delta z_i| \to 0} \sum_{i=0}^{n} w(z_i)\Delta z_i = \int_C w(z)dz \qquad (\text{where } \Delta z_i \to 0 \Rightarrow n \to \infty)$$

here $\Delta z_i = z(t_{i+1}) - z(t_i)$ where we have broken up $t \in [t_A, t_B]$.

Now if we write $w(z) = u(x,y) + iv(x,y)$ and $dz = dx + idy$ then:

$$\int_C w(z)dz = \int_C [u(x,y) + iv(x,y)][dx + idy] = \int_C (udx - vdy) + i\int_C (udy + vdx)$$

Don't mention in class

Using the C parameter t, $dx = \frac{dx}{dt}dt$ and $dy = \frac{dy}{dt}dt$ then:

$$\int_C w(z)dz = \int_{t_A}^{t_B} \left(u\frac{dx}{dt} - v\frac{dy}{dt}\right)dt + i\int_{t_A}^{t_B} \left(u\frac{dy}{dt} + v\frac{dx}{dt}\right)dt$$

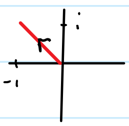
Examples:

1. Let's do $I = \int_C w(z)dz$ w/ $w(z) = z$ along a couple of contours.

Since $w(z) = z = x + iy \Rightarrow u = x \quad v = y$

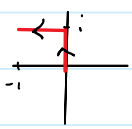
$$\int_c w(z)dz = \int_c (xdx - ydy) + i\int_c (xdy + ydx)$$

a) Let's begin w/ a straight line path from $z=0$ to $z=-1+i$.

Now the contour is: 

so $x = -y$ ($dx = -dy$) and $x_i = 0$, $x_f = -1$

Then: $\int_C w(z)dz = \int_0^{-1}(xdx - xdx) + i\int_0^{-1} -xdx - xdx = -i$

b) Changing the contour to 

so at first $\overbrace{x = 0 \Rightarrow dx = 0}^{y_i = 0 \quad y_f = 1}$, then $\overbrace{y = 1 \Rightarrow dy = 0}^{x_i = 0 \quad x_f = -1}$

Then: $\int_{C'} w(z)dz = \int_0^1 -ydy + \int_0^{-1} xdx + i\int_0^{-1} dx = -i$

These are the same, and in fact for $w(z) = z$ (which is entire) it is completely path independent. Which means of course that if we consider <u>any</u> closed path $\oint_c w(z)dz = 0$.

3. Consider $I = \int_c z^* dz$ w/ the same two contours. Note $w(z) = x - iy \Rightarrow u = x, v = -y$

a) $\int_c w(z)dz = \int_c (xdx + ydy) + i\int_c (xdy - ydx) = \int_0^{-1}(xdx + xdx) + i\int_0^{-1}(-xdx + xdx)$
$= 1$

b) $\int_{c'} w(z)dz = \int_0^1 ydy + \int_0^{-1} xdx - i\int_0^{-1} dx = 1 + i$

Note that w/ the nowhere analytic function $w(z) = z^*$, the value of the integral depends on the path, hence $\oint_c z^* dz \neq 0$ in general.

So here we go:

Cauchy's Theorem: If a function $w(z)$ is analytic within and on a closed contour C, "and $w'(z)$ is continuous throughout this region", then $\oint_C w(z)dz = 0$.

but this can actually be strengthened to:

Cauchy-Goursat Theorem: If a function $w(z)$ is analytic within and on a closed contour C, ~~"and $w'(z)$ is continuous throughout this region"~~, then $\oint_C w(z)dz = 0$.

This comes about because it turns out that functions which are analytic in a region always have continuous derivatives, of all orders, in that region!

Cauchy-Goursat (CG) is one of the most powerful aspects of analytic functions.
Here are some results.

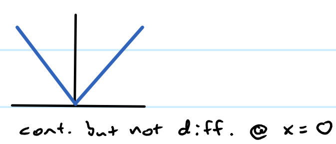
1. Path Independence: As we saw in a couple of examples, integration of analytic functions are path (contour) independent. But this can be proven by the $\oint_C w(z)dz = 0$ of CG:
Consider the integral of an analytic $w(z)$ over two different paths C_1 and C_2 which share the same endpoints. Reversing one $(-C_2)$ and adding them gives a closed path around which $\int_{C_1} w(z)dz + \int_{-C_2} w(z)dz = 0 \Rightarrow \int_{C_1} w(z)dz = -\int_{-C_2} w(z)dz = \int_{C_2} w(z)dz$

2. Fundamental Theorem of Calculus: Remember this? If $f(x) = \int_{x_0}^{x} g(x')dx' \Rightarrow g(x) = \frac{df(x)}{dx}$
i.e. integrals are opposites to derivatives
Well from CG: If $w(z) = \int_{z_0}^{z} g(z')dz' \Rightarrow g(z) = \frac{dw(z)}{dz}$

3. Cauchy's Integral Formula (CIF):
If $w(z)$ is analytic within and on a closed contour C, then for any point z_0 interior to C we get $w(z_0) = \frac{1}{2\pi i}\oint_C \frac{w(z)}{z-z_0}dz$.
In words: If a function is analytic over the region containing the closed curve C, then the values of the function at interior points depend only on the values on the boundary C.

4. Derivatives of Analytic Functions: Using the CIF one can show that all derivatives of any analytic function are also analytic.
In fact: $w'(z_0) = \frac{dw(z_0)}{dz_0} = \frac{1}{2\pi i}\oint_C \frac{w(z)}{(z-z_0)^2}dz$
$w^n(z_0) = \frac{d^n w(z_0)}{dz_0^n} = \frac{n!}{2\pi i}\oint_C \frac{w(z)}{(z-z_0)^{n+1}}dz$

Note this is not true for real functions: $f(x) = x|x| \Rightarrow f'(x) = 2|x|$



5. Liouville's Theorem: [Think about sin(x) vs. sin(z)]
If $w(z)$ is entire and $|w(z)|$ is bounded over all of $z \Rightarrow w(z)$ is a constant.