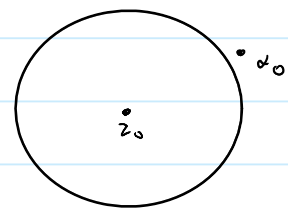
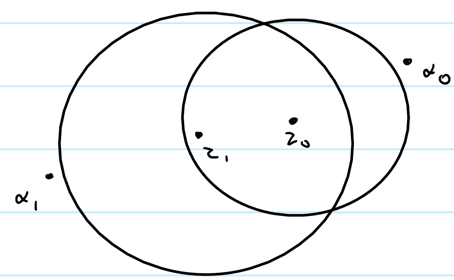
Clearly if a function is analytic over the entire z-plane, then starting w/ any point z_0, we can Taylor expand and by keeping enough terms we get as much accuracy as we want at <u>any</u> other point in z.

An interesting part of the Taylor series story is "analytic continuation". Suppose you were given a function at a point z_0 with a singularity α_0 nearby. This means that you could Taylor expand with precision for a region around z_0 w/ a slightly smaller radius than $|z_0 - \alpha_0|$.



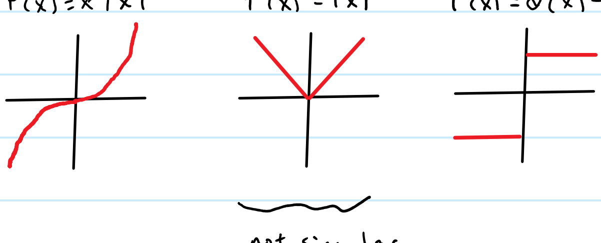
But then you can pick a different point z_1 in this circle and Taylor expand to achieve an accurate description out to its radius of convergence. This may be stopped by yet another singularity α_1.



Clearly continuing this, we will be able to cover the entire $\mathbb{C}$ aside from whatever singularities are there. That is, wherever a function is analytic, we can represent it by the (also analytic) power series w/ positive exponents.

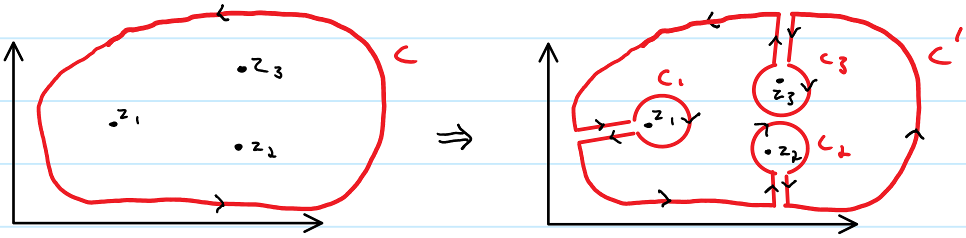
That we can do this with any analytic function is tied to the fact that all of their derivatives always exist, hence we can get arbitrary accuracy in the series representation.

For example: $f(x) = x|x|$ $\quad f'(x) = |x|$ $\quad f''(x) = \theta(x) - \theta(-x)$ $\quad f'''(x)$ - no exist



The integral of $f(z)$ around a closed C containing a finite number n of singular points is equal to the sum of integrals around n circles, each containing a single singular point.

Consider:



$$\oint_C w(z)\,dz \neq 0 \qquad\qquad \oint_{C'} w(z)\,dz = 0$$

Now if we take the two straight paths leading to each small circle very close, then since $L_1 = -L_2$ we have $\int_{L_1} w(z)\,dz + \int_{L_2} w(z)\,dz = 0$, and this also completes continuity around C, C_1, C_2, C_3 giving:

$$\oint_{C'} w(z)\,dz = 0 = \oint_C w(z)\,dz + \oint_{C_1} w(z)\,dz + \oint_{C_2} w(z)\,dz + \oint_{C_3} w(z)\,dz$$

this means integrate counterclockwise

$$\Rightarrow \oint_C w(z)\,dz = 2\pi i \sum_{j=1}^{n} R_j \quad \text{where } R_j = \frac{1}{2\pi i}\oint_{C_j} w(z)\,dz = \text{residue at point } z_j$$

Example:

$$I = \oint_C \frac{3z^2+2}{z(z+1)}\,dz \quad \text{w/ } C: |z| = 3$$

then

$$I = \oint_{C_0} \frac{(3z^2+2)/(z+1)}{z}\,dz + \oint_{C_1} \frac{(3z^2+2)/z}{z+1}\,dz$$

where C_0 (C_1) encloses 0 (-1) but not -1 (0).

Note each integral above is $\oint_C \frac{w(z)}{z-z_0}\,dz$ where $w(z)$ is analytic within C due to

Thus using CIF $w(z_0) = \frac{1}{2\pi i}\oint_C \frac{w(z)}{z-z_0}\,dz$ we can evaluate each one.

$$\oint_{C_0} \frac{(3z^2+2)/(z+1)}{z}\,dz = 2\pi i\left(\frac{2}{1}\right) = 4\pi i$$

$$\oint_{C_1} \frac{(3z^2+2)/z}{z+1}\,dz = 2\pi i\left(\frac{5}{-1}\right) = -10\pi i$$

$$\Bigg\}\; I = -6\pi i$$

Another example:

$$I = \oint_C \frac{3z+2}{z(z+1)^2}\,dz \quad \text{w/ } C: |z| = 3$$

then

$$I = \oint_{C_0} \frac{(3z+2)/(z+1)^2}{z}\,dz + \oint_{C_1} \frac{(3z+2)/z}{(z+1)^2}\,dz = \oint_{C_0} \frac{w_0(z)}{z}\,dz + \oint_{C_1} \frac{w_1(z)}{(z+1)^2}\,dz$$

We can use CIF immediately on the first giving: $\oint_{C_0} \frac{w_0(z)}{z}\,dz = 2\pi i\, w_0(0) = 4\pi i$

However for the second integral we need to massage:

$$\oint_{C_1} \frac{w_1(z)}{(z+1)^2}\,dz = 2\pi i \left.\frac{dw_1(z_0)}{dz_0}\right|_{z_0=-1} \quad \text{from the CIF route to derivatives}$$

$$= 2\pi i \left[\frac{d}{dz_0}\left(3 + \frac{2}{z_0}\right)\right]_{z_0=-1}$$

$$= 2\pi i \left(-\frac{2}{z_0^2}\right)_{z_0=-1}$$

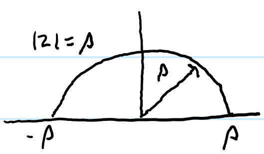
$$= -4\pi i$$

So in total: $I = 0$

In fact this example uses the useful result from using the CIF to find derivatives.

That is: $\frac{f^{n-1}(z_0)}{(n-1)!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^n}\,dz$

We'll close w/ a useful result know as Jordan's Lemma.



If as $\rho \to \infty$, $|R(z)| \to 0$ uniformly in θ for $0 < \theta < \pi$, then
$\lim_{\rho\to\infty} \int_{|z|=\rho} R(z) e^{i\alpha z}\,dz = 0$ for $\alpha > 0$

Proof: Let $M(\rho)$ be the maximum value of $|R(z)|$ on the semi-circle $|z| = \rho$. Then the statement that $|R(z)| \to 0$ uniformly means that $|R(z)| \le M(\rho)$ where $\lim_{\rho\to\infty} M(\rho) = 0$ ind. of θ.

Thus if $I = \int_{|z|=\rho} R(z) e^{i\alpha z}\,dz$

$$|I| \le M(\rho) \int_0^\pi \left|e^{i\alpha(\rho\cos\theta + i\rho\sin\theta)}\right| \left|\rho e^{i\theta}\right| d\theta$$

$$= M(\rho) \int_0^\pi e^{-\frac{i}{2}\alpha(\rho\cos\theta - i\rho\sin\theta)} e^{\frac{i}{2}\alpha(\rho\cos\theta + i\rho\sin\theta)} \rho\, e^{-i\theta} e^{i\theta}\, d\theta$$

$$= M(\rho) \int_0^\pi e^{i\alpha\rho\sin\theta} \rho\, d\theta$$

$$= M(\rho)\,\pi\, \frac{(1 - e^{-\alpha\rho})}{\alpha} \to 0 \quad \text{as } \rho \to \infty \text{ since } \alpha > 0.$$

Green's functions are a means of transforming differential equations into integral equations.

Sometimes this is easy: $\frac{dy}{dx} = f(x)$ w/ $y(a) = y_0 \Rightarrow dy = f(x)dx \Rightarrow y(x) = y_0 + \int_a^x f(x')dx'$

Sometimes this is not: $\frac{dy}{dx} = f(x,y)$ w/ $y(a) = y_0 \Rightarrow ?$ or $\frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} = f(x) \Rightarrow ?$

Starting out quite generally, consider $Ly = f$ where L is a linear ordinary differential operator, i.e. $\alpha_0(x)y + \alpha_1(x)\frac{dy}{dx} + \alpha_2(x)\frac{d^2y}{dx^2} + \dots + \alpha_n(x)\frac{d^ny}{dx^n}$, and f is a given function of x.

Suppose that L possesses a complete and orthonormal set of eigenfunctions: $L\phi_n(x) = \lambda_n\phi_n(x)$

Because they are complete, we know: $y(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n(x)$ and $f(x) = \sum_{n=1}^{\infty} \beta_n \phi_n(x)$

Shoving these in:

$$Ly = L\sum_{n=1}^{\infty} \alpha_n \phi_n(x) = \sum_{n=1}^{\infty} L\alpha_n\phi_n = \sum_{n=1}^{\infty} \alpha_n\lambda_n\phi_n = f = \sum_{n=1}^{\infty} \beta_n\phi_n(x) \Rightarrow \sum_{n=1}(\alpha_n\lambda_n - \beta_n)\phi_n = 0$$

But since ϕ_n are lin. ind. $\Rightarrow \alpha_n\lambda_n - \beta_n = 0 \Rightarrow \alpha_n = \frac{\beta_n}{\lambda_n} \Rightarrow y(x) = \sum_{n=1}^{\infty} \frac{\beta_n}{\lambda_n}\phi_n(x)$

Now recall that $\beta_n = \overbrace{(\phi_n, f)}^{\int_a^b \phi_n^*(x')f(x')dx' \text{ where } a,b \text{ are bounds of } x,x'} \Rightarrow y(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n}\phi_n(x)(\phi_n, f) = \int_a^b \underbrace{\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n}}_{G(x,x') \text{ - Green's Function}} f(x')dx'$

Consider: $LG(x,x') = \sum_{n=1}^{\infty} \frac{L\phi_n(x)\phi_n^*(x')}{\lambda_n} = \sum_{n=1}^{\infty} \phi_n(x)\phi_n^*(x') \equiv I(x,x')$ for L

But: $\underbrace{\int_a^b I(x,x')f(x')dx}_{} = \sum_{n=1}^{\infty}\phi_n(x)\int_a^b \phi_n^*(x')f(x')dx' = \sum_{n=1}^{\infty}\phi_n(x)(\phi_n, f) = \underline{f(x)}$

Thus: $I(x,x') = \delta(x-x') \Rightarrow LG(x,x') = \delta(x-x')$ Now squint: $\underline{LG = I \Rightarrow G \simeq L^{-1}}$

(The interpretation of Green's functions)

So we have for $Ly = f$: $y = L^{-1}f = \int_a^b G(x,x')f(x')dx'$

Another way this is written is: $LK = I$ w/ $Kf = \int G(x,x')f(x')dx'$

note this the identity, not $I(x,x')$

Let's view this in a simple example:

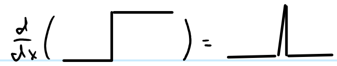
$$L = \frac{d}{dx} \Rightarrow Ly = f \text{ w/ } y(a) = y_0 \Rightarrow y(x) = y_0 + \overbrace{\int_a^b G(x,x')f(x')dx'}^{=0 \text{ for } x=a}$$

compare to

$$\frac{d}{dx}y = f \text{ w/ } y(a) = y_0 \Rightarrow \underset{x \in [a,b]}{y(x)} = y_0 + \int_a^x f(x')dx' = y_0 + \int_a^b \underbrace{\theta(x-x')}\, f(x')dx'$$

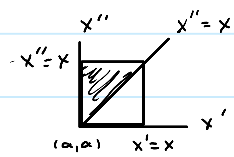
$$\theta(x-x') = \begin{cases} 1 & x > x' \\ 0 & x \le x' \end{cases}$$

Thus: $G(x,x') = \theta(x-x')$, but recall that $LG(x,x') = \delta(x-x') \Rightarrow \frac{d\theta(x-x')}{dx} = \delta(x-x')$



Let's try another:

$$L = \frac{d^2}{dx^2} \Rightarrow Ly = f \text{ w/ } y(a) = y_0,\ y'(a) = \hat{y}_0 \Rightarrow \frac{dy}{dx} = \hat{y}_0 + \int_a^x f(x')dx'$$



$$\begin{aligned} y(x) &= y_0 + \hat{y}_0(x-a) + \int_a^x dx'' \int_a^{x''} dx' f(x') \\ &= y_0 + \hat{y}_0(x-a) + \int_a^x f(x')dx' \int_{x'}^x dx'' \\ &= y_0 + \hat{y}_0(x-a) + \int_a^x (x-x')f(x')dx' \\ &= y_0 + \hat{y}_0(x-a) + \int_a^b \underbrace{(x-x')\theta(x-x')}_{G(x,x')} f(x')dx' \end{aligned}$$

And since $LG(x,x') = \delta(x-x') \Rightarrow \frac{d^2}{dx^2}\left[(x-x')\theta(x-x')\right] = \frac{d}{dx}\left[\theta(x-x') + \underbrace{(x-x')\delta(x-x')}_{=0}\right]$

$$= \delta(x-x')$$

And one more:

$$L = \frac{d^2}{dx^2} \Rightarrow Ly = f \text{ w/ } y(0) = y_0,\ y(1) = y_1 \Rightarrow y(x) = \alpha + \beta x + \int_0^1 (x-x')\theta(x-x')f(x')dx'$$

find α, β from boundary conditions

$$= y_0 + (y_1 - y_0)x - \int_0^1 x(1-x')f(x')dx' + \int_0^1 (x-x')\theta(x-x')f(x')dx'$$

So $G(x,x') = -x(1-x') + (x-x')\theta(x-x') = \begin{cases} -x'(1-x) & 0 \le x' \le x \quad (\theta = 1) \\ -x(1-x') & x \le x' \le 1 \quad (\theta = 0) \end{cases}$

And once again: $LG(x,x') = \delta(x-x')$