In 3D things get harder, so it makes sense that we are relegated to a smaller class of operators. Fortunately, these include physically relevant ones.

Consider $H_0 = \nabla^2 + \lambda$ w/ $\nabla^2\phi = \nabla\cdot(\nabla\phi) = \frac{1}{h_1h_2h_3}\left[\partial_1\left(\frac{h_2h_3}{h_1}\partial_1\phi\right) + \partial_2\left(\frac{h_3h_1}{h_2}\partial_2\phi\right) + \partial_3\left(\frac{h_1h_2}{h_3}\partial_3\phi\right)\right]$

where $ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2$ [see Table 1.1 in book]

It turns out that the Fourier transform method used last time works here as well.

So consider: $H_0\phi(\vec{r}) = F(\vec{r})$ w/

$$\hat{\phi}(\vec{k}) = \frac{1}{(2\pi)^{3/2}}\int e^{-i\vec{k}\cdot\vec{r}}\phi(\vec{r})\,d^3\vec{r}$$
$$\hat{F}(\vec{k}) = \frac{1}{(2\pi)^{3/2}}\int e^{-i\vec{k}\cdot\vec{r}}F(\vec{r})\,d^3\vec{r}$$

Then: $\frac{1}{(2\pi)^{3/2}}\underbrace{\int e^{-i\vec{k}\cdot\vec{r}}\nabla^2\phi(\vec{r})\,d^3\vec{r}} + \lambda\hat{\phi}(\vec{k}) = \hat{F}(\vec{k})$

Now recall that in 1D, we just IBP until we could combine this w/ $\hat{\phi}$.

Well, we can do that in 3D using Green's theorem:

$\int_V (F\nabla^2 G - G\nabla^2 F)\,d^3\vec{r} = \int_S (F\nabla G - G\nabla F)\cdot\hat{n}\,dS$ where S encloses V and $\hat{n}$ is the outward unit normal to S

Then:

$$\frac{1}{(2\pi)^{3/2}}\int_V e^{-i\vec{k}\cdot\vec{r}}\nabla^2\phi(\vec{r})\,d^3\vec{r} = \frac{1}{(2\pi)^{3/2}}\int_V \phi(\vec{r})\nabla^2 e^{-i\vec{k}\cdot\vec{r}}\,d^3\vec{r} + \frac{1}{(2\pi)^{3/2}}\underbrace{\int_S\left[e^{-i\vec{k}\cdot\vec{r}}\nabla\phi(\vec{r}) - \phi(\vec{r})\nabla e^{-i\vec{k}\cdot\vec{r}}\right]\cdot\hat{n}\,dS}$$

Now what is the volume V? It is the entirety of $\mathbb{R}^3$.

So what is S? A sphere of radius $R\to\infty$ w/ $\hat{n} = \hat{r}$ $\Rightarrow \to 0$ if $\phi(\vec{r})\to 0$ faster than $\frac{1}{r}$

Then: $\frac{1}{(2\pi)^{3/2}}\int_V e^{-i\vec{k}\cdot\vec{r}}\nabla^2\phi(\vec{r})\,d^3\vec{r} = -k^2\hat{\phi}(\vec{k})$

So: $(-k^2+\lambda)\hat{\phi}(\vec{k}) = \hat{F}(\vec{k})$

There are two important cases to consider: a) $\lambda < 0$

b) $\lambda \geq 0$

For $\underline{\underline{\lambda < 0}}$ call $\lambda = -n^2$ and $\hat{\phi}(\vec{k}) = -\dfrac{\hat{F}(\vec{k})}{\underbrace{k^2+n^2}_{\neq 0}}$

$$\hat{F}(\vec{k}) = \frac{1}{(2\pi)^{3/2}}\int F(\vec{r}\,')e^{-i\vec{k}\cdot\vec{r}\,'}d^3\vec{r}\,'$$

Now we start undoing: $\phi(\vec{r}) = \underbrace{\lambda(\vec{r})} - \dfrac{1}{(2\pi)^{3/2}}\displaystyle\int \frac{\hat{F}(\vec{k})}{k^2+n^2}e^{i\vec{k}\cdot\vec{r}}d^3\vec{k}$

solution to $H_0\lambda(\vec{r}) = (\nabla^2 - n^2)\lambda(\vec{r}) = 0$

but none exist that $\to 0$ as $|\vec{r}|\to\infty$

So: $\phi(\vec{r}) = \displaystyle\int G(\vec{r},\vec{r}\,')F(\vec{r}\,')d^3\vec{r}\,'$ w/ $G(\vec{r},\vec{r}\,') = -\dfrac{1}{(2\pi)^3}\displaystyle\int \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{k^2+n^2}d^3\vec{k}$

But: $I = \displaystyle\int \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{k^2+n^2}d^3\vec{k} = \int_0^{2\pi}d\phi\int_0^{\pi}\underbrace{\sin\theta\, d\theta}_{\cos\theta = t}\int_0^\infty k^2\frac{e^{ik|\vec{r}-\vec{r}\,'|\cos\theta}}{k^2+n^2}dk$ (using k,θ,ϕ)

$\cos\theta = t \Rightarrow -\sin\theta\, d\theta = dt$ w/ $t_i = 1$, $t_f = -1$

$$= -2\pi\int_1^{-1}dt\int_0^\infty k^2\frac{e^{ik|\vec{r}-\vec{r}\,'|t}}{k^2+n^2}dk$$

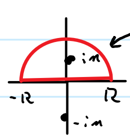
$$= \frac{2\pi}{i|\vec{r}-\vec{r}\,'|}\int_0^\infty\left(\underbrace{ke^{ik|\vec{r}-\vec{r}\,'|}}-\underbrace{ke^{-ik|\vec{r}-\vec{r}\,'|}}\right)\frac{dk}{k^2+n^2}$$

over $k\in[0,\infty)$ is just like this over $k\in[0,-\infty)$

$$= \frac{2\pi}{i|\vec{r}-\vec{r}\,'|}\underbrace{\int_{-\infty}^{\infty}\frac{ke^{ik|\vec{r}-\vec{r}\,'|}}{k^2+n^2}dk}_{\text{part of}}$$

part of $\displaystyle\oint\frac{ke^{ik|\vec{r}-\vec{r}\,'|}}{\underbrace{k^2+n^2}_{(k+in)(k-in)}}dk$ around

And again due to Jordan's lemma semicircle $= 0$ w/ $R\to\infty$



$$= \frac{2\pi}{i|\vec{r}-\vec{r}\,'|}2\pi i\, R_{in}$$

$$R_{in} = \left[\frac{ke^{ik|\vec{r}-\vec{r}\,'|}}{k+in}\right]_{k=in}$$

$$= \frac{4\pi^2}{|\vec{r}-\vec{r}\,'|}\frac{in\,e^{-n|\vec{r}-\vec{r}\,'|}}{in+in} = \frac{2\pi^2}{|\vec{r}-\vec{r}\,'|}e^{-n|\vec{r}-\vec{r}\,'|}$$

Thus: $G(\vec{r},\vec{r}\,') = -\dfrac{1}{4\pi}\dfrac{e^{-n|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}$ (You should have seen this before!)

And finally: $\phi(\vec{r}) = -\dfrac{1}{4\pi}\displaystyle\int\frac{e^{-n|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}F(\vec{r}\,')d^3\vec{r}\,'$

Now as we take $|\vec{r}|\to\infty$, if $F(\vec{r}\,')$ decreases rapidly enough (say for $|\vec{r}\,'|>R$ then $F(\vec{r}\,')=0$) then integration over dr' will give 0 beyond a finite value of $|\vec{r}\,'| = R$, and hence as $|\vec{r}|\to\infty$ it will dominate against $|\vec{r}\,'|$ leaving $\phi(\vec{r})\longrightarrow \underbrace{C}\dfrac{e^{-nr}}{r}$ [C involves angular parts]

Note: $\phi\to 0$ faster than $\frac{1}{r}$!

For $\underline{\underline{\lambda \geq 0}}$ we know that $-k^2+\lambda=0 \Rightarrow k=\pm\sqrt{\lambda}$ so our usual algebraic manipulation might not hold. Now this is of course due to an integration over k along the real axis which encounters these points as singular. But suppose they were actually complex.

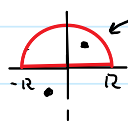
Suppose $\lambda = (q \pm i\epsilon)^2$ w/ $\epsilon > 0$

Then the usual steps: $\hat{\phi}_{\pm}(\vec{k}) = -\underbrace{\dfrac{\hat{F}(\vec{k})}{k^2-(q\pm i\epsilon)^2}}_{\neq 0 \text{ for real } k} \Rightarrow G_{\pm}(\vec{r},\vec{r}\,') = -\dfrac{1}{(2\pi)^3}\underbrace{\int \dfrac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{k^2-(q\pm i\epsilon)^2}\, d^3\vec{k}}_{I_{\pm}}$

Repeating our previous efforts: $$I_{\pm} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\infty} k^2 \frac{e^{ik|\vec{r}-\vec{r}\,'|\cos\theta}}{k^2-(q\pm i\epsilon)^2}\, dk$$

(The book gets stupid here!)

$$= \frac{2\pi}{i|\vec{r}-\vec{r}\,'|}\int_{-\infty}^{\infty} k \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{(k-q\mp i\epsilon)(k+q\pm i\epsilon)}\, dk$$

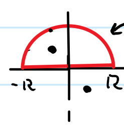
We can evaluate I_+ as part of the closed contour integral:  w/ $R \to \infty$

And again due to Jordan's lemma semicircle = 0

$$I_+ = \frac{2\pi}{i|\vec{r}-\vec{r}\,'|} 2\pi i \left[k \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{(k+q+i\epsilon)} \right]_{k=q+i\epsilon}$$

$$= \frac{4\pi^2}{|\vec{r}-\vec{r}\,'|} \frac{(q+i\epsilon)e^{iq|\vec{r}-\vec{r}\,'|}e^{\epsilon|\vec{r}-\vec{r}\,'|}}{2(q+i\epsilon)}$$

$$= 2\pi^2 \frac{e^{iq|\vec{r}-\vec{r}\,'|}e^{\epsilon|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}$$

And I_- as part of the closed contour integral:  w/ $R \to \infty$

And again due to Jordan's lemma semicircle = 0

$$I_- = \frac{2\pi}{i|\vec{r}-\vec{r}\,'|} 2\pi i \left[k \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{(k-q+i\epsilon)} \right]_{k=-q+i\epsilon}$$

$$= 2\pi^2 \frac{e^{-iq|\vec{r}-\vec{r}\,'|}e^{\epsilon|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}$$

Then $G_{\pm}(\vec{r},\vec{r}\,') = -\dfrac{1}{4\pi}\dfrac{e^{\pm iq|\vec{r}-\vec{r}\,'|}e^{\epsilon|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}$

With $G_\pm(\vec{r},\vec{r}')$ in hand we then have:

$$\phi_\pm(\vec{r}) = \underbrace{\lambda(\vec{r})}_{} - \frac{1}{4\pi}\int \frac{e^{i(\pm q + i\epsilon)|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} F(\vec{r}')\, d^3\vec{r}'$$

solution to $H_0 \lambda(\vec{r}) = (\nabla^2 + q^2)\lambda(\vec{r}) = 0 \;\Rightarrow\; \lambda(\vec{r}) = \frac{A}{(2\pi)^{3/2}} e^{i\vec{q}\cdot\vec{r}}$

$$\phi_\pm(\vec{r}) = \frac{A}{(2\pi)^{3/2}} e^{i\vec{q}\cdot\vec{r}} - \frac{1}{4\pi}\int \frac{e^{i(\pm q + i\epsilon)|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} F(\vec{r}')\, d^3\vec{r}'$$

And now for $|\vec{r}| \to \infty$, assuming $F(\vec{r}') \to 0$ fast enough and taking $\epsilon \to 0$,

$$\phi_\pm(\vec{r}) \longrightarrow \frac{A}{(2\pi)^{3/2}} e^{i\vec{q}\cdot\vec{r}} - \frac{1}{4\pi}\frac{e^{\pm iqr}}{r}$$

Note very different asymptotic behavior for $\lambda < 0$ and $\lambda \geq 0$.

So let's apply these 3D results to physics. Recall $H_0\phi(\vec{r}) = (\nabla^2+\lambda)\phi(\vec{r}) = F(\vec{r})$

Now the simple thought is that given $F(\vec{r})$ we use Green's functions to find $\phi(\vec{r})$, but...

Consider the time independent Schrodinger equation:

$$H\phi(\vec{r}) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right]\phi(\vec{r}) = E\,\phi(\vec{r}) \Rightarrow (\nabla^2 + \overbrace{\frac{2mE}{\hbar^2}}^{\lambda})\phi(\vec{r}) = \frac{2m}{\hbar^2}V(\vec{r})\phi(\vec{r})$$

Now recall that for a repulsive potential $V(\vec{r})>0$ and we can only have scattering states w/ $E>0$. However for an attractive potential $V(\vec{r})<0$ and we can have both scattering states w/ $E>0$ and bound states w/ $E<0$. Note scattering vs. bound determines which solution since $\mathrm{sgn}(E)=\mathrm{sgn}(\lambda)$.

Starting w/ bound states or $\lambda<0$ for which $\phi(\vec{r}) = -\frac{1}{4\pi}\int \frac{e^{-\kappa|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}F(\vec{r}\,')d^3\vec{r}\,'$

So $\lambda = -\kappa^2 = \frac{2mE}{\hbar^2}$ and we then have:

$$\phi(\vec{r}) = -\frac{1}{4\pi}\int \frac{e^{-\kappa|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\underbrace{\frac{2m}{\hbar^2}V(\vec{r}\,')\,\phi(\vec{r}\,')}_{F(\vec{r})}d^3\vec{r}\,'$$

Consider the Yukawa potential $V(\vec{r}) = -g^2\frac{e^{-\mu r}}{r}$, then:

$$\phi(\vec{r}) = \frac{mg^2}{2\pi\hbar^2}\int \frac{e^{-\kappa|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\frac{e^{-\mu r'}}{r'}\phi(\vec{r}\,')\,d^3\vec{r}\,'$$ This is called a linear integral eigenvalue problem.

Now let's consider scattering states for which $\lambda \geq 0$ and taking $\epsilon\to 0$:

$$\phi_\pm(\vec{r}) = \frac{A}{(2\pi)^{3/2}}e^{i\vec{q}_i\cdot\vec{r}} - \frac{m}{2\pi\hbar^2}\int \frac{e^{\pm i q_i|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}V(\vec{r}\,')\,\phi_\pm(\vec{r}\,')d^3\vec{r}\,' \quad \text{where } q_i = \sqrt{\frac{2mE}{\hbar^2}}$$

Let's consider the $r\to\infty$ limit.

$$|\vec{r}-\vec{r}\,'| = (r^2 - 2\vec{r}\cdot\vec{r}\,' + r'^2)^{1/2} = r\left[1-\frac{2\vec{r}\cdot\vec{r}\,'}{r^2}+\frac{r'^2}{r^2}\right]^{1/2} = r\left[1-\frac{2\hat{n}\cdot\vec{r}\,'}{r}+\frac{r'^2}{r^2}\right]^{1/2}$$
$$= r - \hat{n}\cdot\vec{r}\,' + \mathcal{O}\left(\tfrac{1}{r}\right)$$

Then:

$$\phi_\pm(\vec{r}) \longrightarrow \frac{A}{(2\pi)^{3/2}}e^{i\vec{q}_i\cdot\vec{r}} - \frac{m}{2\pi\hbar^2}\frac{e^{\pm i q_i r}}{r}\int e^{\mp i q_i\hat{n}\cdot\vec{r}\,'}V(\vec{r}\,')\,\phi_\pm(\vec{r}\,')\,d\vec{r}\,'$$

In both cases the book got silly. It realized that the given integrals are hard, and so it decided to Fourier transform these results to reach slightly easier integrals relating $\hat{\phi}(\vec{k})$ to itself. But it could have just Fourier transformed the Schrodinger equation itself and skipped the Green's function use!

So keeping in the spirit of Green's functions, we'll press on.

Let's consider extending this story to 4D and consider the wave-equation:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\vec{r},t) = -\frac{\rho(\vec{r})}{\epsilon_0}$$

Now the Green's function relevent for this should satisfy:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\vec{r},t,\vec{r}_0,t_0) = \delta(\vec{r}-\vec{r}_0)\,\delta(t-t_0)$$

Now let's Fourier transform the time part of this:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\underbrace{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\vec{r},t,\vec{r}_0,t_0)e^{i\omega t}}_{\text{playing the I.B.P. game}}\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(\vec{r}-\vec{r}_0)\,\delta(t-t_0)e^{i\omega t}\,dt$$

$$\frac{1}{\sqrt{2\pi}}\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\hat{G}(\vec{r},\vec{r}_0,\omega) = \frac{1}{\sqrt{2\pi}}\delta(\vec{r}-\vec{r}_0)e^{i\omega t_0}$$

or

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\hat{G}(\vec{r},\vec{r}_0,\omega) = \delta(\vec{r}-\vec{r}_0)e^{i\omega t_0}$$

Now recall that for $(\nabla^2 + \lambda^2)\hat{G}(\vec{r},\vec{r}_0) = \delta(\vec{r}-\vec{r}_0) \Rightarrow G_{\pm}(\vec{r},\vec{r}_0) = -\frac{1}{4\pi}\frac{e^{\pm i\lambda|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|}$

So perhaps: $\hat{G}_{\pm}(\vec{r},\vec{r}_0,\omega) = -\frac{1}{4\pi}\frac{e^{\pm i\frac{\omega}{c}|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|}e^{i\omega t_0}$ (in fact this works!)

Fourier transforming back we have:

$$G_{\pm}(\vec{r},t,\vec{r}_0,t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} -\frac{1}{4\pi}\frac{e^{\pm i\frac{\omega}{c}|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|}e^{-i\omega(t-t_0)}\,d\omega$$

$$= -\frac{1}{4\pi|\vec{r}-\vec{r}_0|}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega\left[\mp\frac{|\vec{r}-\vec{r}_0|}{c}+(t-t_0)\right]}\,d\omega$$

$$= -\frac{\delta\left([t-t_0]\mp\frac{|\vec{r}-\vec{r}_0|}{c}\right)}{4\pi|\vec{r}-\vec{r}_0|}$$

Therefore: $\phi_{\pm}(\vec{r},t) = \phi_0(\vec{r},t) + \int\frac{\delta\left([t-t_0]\mp\frac{|\vec{r}-\vec{r}_0|}{c}\right)}{4\pi|\vec{r}-\vec{r}_0|}\frac{\rho(\vec{r}_0)}{\epsilon_0}\,d\vec{r}_0$

The δ-function means that nonzero contributions from $\rho(\vec{r}_0)$ at $\vec{r},t$ will only come from $t-t_0 \mp \frac{|\vec{r}-\vec{r}_0|}{c} = 0 \Rightarrow |r-r_0| = \pm c(t-t_0)$, that is, things that are "light-distance" away.