The standard language is that A' is "similar" to A.

Let's see this play out in an example.

Recall $D = \frac{d}{dt}$ on P_1 w/ $x = \alpha_0 + \alpha_1 t = \frac{1}{2}(\alpha_0+\alpha_1)(1+t) + \frac{1}{2}(\alpha_0-\alpha_1)(1-t)$.

In the $\{1, t\}$ basis $D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\alpha = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \Rightarrow D\alpha = \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix}$ or $y = \alpha_1$

In the $\{1+t, 1-t\}$ basis $D' = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$, $\alpha' = \begin{pmatrix} \frac{1}{2}(\alpha_0+\alpha_1) \\ \frac{1}{2}(\alpha_0-\alpha_1) \end{pmatrix} \Rightarrow D'\alpha' = \begin{pmatrix} \frac{1}{2}\alpha_1 \\ \frac{1}{2}\alpha_1 \end{pmatrix}$ or $y' = \alpha_1$

So what is the matrix that changes the basis $\{1, t\}$ to $\{1+t, 1-t\}$, or rather the matrix which reflects how this acts on components?

It's just $\underbrace{B^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}_{\text{recall that since } B \text{ acts on the basis, } B^{-1} \text{ acts on the components}} \Rightarrow B^{-1}\alpha = \alpha'$ and $B^{-1} D B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = D'$

So in terms of the matrix action on the components only we have:

$$D\alpha = B B^{-1} D B B^{-1} \alpha = B D' \alpha' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{2}\alpha_1 \\ \frac{1}{2}\alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix} \checkmark$$

Okay so we now know that any n-dim vector space over F is isomorphic to F^n, and any lin. trans. on the vector space is isomorphic to an $n \times n$ matrix acting on F^n.

We also know that for finite n, the inverse of a lin. trans. A exists if $Ax = 0 \Rightarrow x = 0$.

Now we would like to cast this condition in the matrix language, since perhaps it is easier to verify (turns out it is!).

But wait... why? Well recall, many "problems" end up as $A\underset{\text{unknown}}{x} = \overset{\text{known}}{y}$, but $x = A^{-1}y$.

Okay so in matrix language it turns out that the determinate plays a crucial role here. So before we go to inverses, lets get this out.

Given a linear transformation A w/ $n \times n$ matrix elements a_{ij}, then:

$\det A = \epsilon_{abc\cdots n}\, a_{1a} a_{2b} \cdots a_{Nn}$ (Einstein summation)

where $\epsilon_{abc\cdots n} = \begin{cases} +1 & \text{if } (abc\cdots n) \text{ is an even permutation of } (12\cdots N) \\ -1 & \text{if } (abc\cdots n) \text{ is an odd permutation of } (12\cdots N) \\ 0 & \text{otherwise, e.g. if any index is repeated} \end{cases}$

Note: $\epsilon_{abc\cdots n}$ is negative if you swap any two side by side elements $\Rightarrow$ its negative if you swap any two

e.g. ϵ_{1234} $\Rightarrow$ exchange $1 \leftrightarrow 4$ $\epsilon_{4231} = -\epsilon_{2431} = \epsilon_{2341} = -\epsilon_{2314} = \epsilon_{2134} = -\epsilon_{1234}$

or $1 \leftrightarrow 3$ $\epsilon_{3214} = -\epsilon_{3124} = \epsilon_{1324} = -\epsilon_{1234}$

So this is a sum/difference of terms. How many? For:

not cyclic $\rightarrow$ $n = 2$ we have $\epsilon_{12} = +1$, $\epsilon_{21} = -1$

cyclic $\rightarrow$ $n = 3$ we have $\epsilon_{123} = +1$, $\epsilon_{312} = +1$, $\epsilon_{231} = +1$, $\epsilon_{213} = -1$, $\epsilon_{132} = -1$, $\epsilon_{321} = -1$

more than cyclic $\rightarrow$ $n = 4$ we have $1234 = 1342 = 1423 = 4213 = 3241 = 4132 = 2431 = 2314 = 3124 = 2143 = 4321 = 3412 = +1$

and w/ $(1 \leftrightarrow 2)$ $2134 = 2341 = 2413 = 4123 = 3142 = 4231 = 1432 = 1324 = 3214 = 1243 = 4312 = 3421 = -1$

So $n!$ nonzero terms.

Note: $n = 2$ $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow \det A = \cancel{\epsilon_{11}} a_{11} a_{21} + \underbrace{\epsilon_{12} a_{11} a_{22} + \epsilon_{21} a_{12} a_{21}}_{\epsilon_{ij} a_{1i} a_{2j} \text{ sums over columns}} + \cancel{\epsilon_{22}} a_{12} a_{22}$

$= + a_{11} a_{22} - a_{12} a_{21}$

$= \cancel{\epsilon_{11}} a_{11} a_{12} + \underbrace{\epsilon_{12} a_{11} a_{22} + \epsilon_{21} a_{21} a_{12}}_{\epsilon_{ij} a_{i1} a_{j2} \text{ sums over rows}} + \cancel{\epsilon_{22}} a_{21} a_{22}$

The interchange of row and column means: $\det A = \epsilon_{abc\cdots n} a_{a1} a_{b2} \cdots a_{nN}$ as well.
Since $r \leftrightarrow c$ doesn't impact $\det A$, then $\det A = \det \tilde{A} = \det (A^{+})^{*}$
This means that any property that arises from doing something to rows also applies to columns.

Some more properties:

1. A common factor of each row or column may be factored out.

$\det \begin{pmatrix} \lambda a & b \\ \lambda c & d \end{pmatrix} = \lambda ad - \lambda bc = \lambda \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ but of course $\begin{pmatrix} \lambda a & b \\ \lambda c & d \end{pmatrix} \neq \lambda \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$

In this case you'd get λ^2.

2. If any row or column is all zeros then $\det A = 0$.
3. $\det \underline{I} = 1$
4. Interchanging two rows or columns changes the sign of the determinant.

$\det A = \underbrace{\epsilon_{ab\cdots n}} \underbrace{A_{1a} A_{2b} \cdots A_{nN}}$

This product commutes so order is not important.

However swapping indices on this always generates a minus.

5. If any two rows or columns are equal then $\det A = 0$.

If you exchange two identical rows or columns, then the resulting matrix looks exactly the same as before, except we're supposed to get $\det A = -\det A'$ (w/ $A = A'$) so $\det A = 0$.

6. $\det(AB) = \det A \det B$ proven using $\epsilon_{abc\cdots n} \det A = \epsilon_{a'b'c'\cdots n'} A_{aa'} A_{bb'} \cdots A_{nn'}$
7. A scaler multiple of a row (or column) may be added to another row (or column) w/out changing $\det A$.

e.g. $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \begin{pmatrix} a+\lambda c & b+\lambda d \\ c & d \end{pmatrix} = \det \begin{pmatrix} a+\lambda b & b \\ c+\lambda d & d \end{pmatrix} = ad - bc$

8. $\det A = 0$ iff the row (or column) "vectors" are linearly dependent

e.g. $\det \begin{pmatrix} a & d & \lambda a + d \\ b & e & \lambda b + e \\ c & f & \lambda c + f \end{pmatrix} = \lambda aec + aef - \lambda afb - afe - \lambda dbc - dbf + \lambda dcb + dce$
$+ \lambda abf - \lambda ace + dbf - dce = 0$

Now we can say something important: [If $\det A = 0 \Rightarrow A^{-1}$ cannot exist

Quick proof: If A^{-1} existed then $\det(A^{-1}A) = \det(I) = 1 = \det A^{-1} \det A = \det A^{-1}\, 0 = 0$

but $1 \neq 0$, so A^{-1} cannot exist. Note that given this A^{-1} might not exist even if $\det A \neq 0$.

But even more important is: [The matrix has an inverse if and only if $\det A \neq 0$.

To prove this requires machinery.

"cofactor"

$$\det A = \sum_j A_{Jj}\,\mathrm{cof}(A_{Jj}) = \sum_j A_{jJ}\,\mathrm{cof}(A_{jJ})$$ for a fixed row or column J.

$\mathrm{cof}(A_{ij})$ is $(-1)^{i+j}$ times the determinant of the matrix resulting by ignoring the i row and j column.

And:

$$\delta_{KJ}\det A = \sum_j A_{Kj}\,\mathrm{cof}(A_{Jj}) = \sum_j A_{jK}\,\mathrm{cof}(A_{jJ})$$

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

If $K=J$ then this reduces to the previous theorem.

Let's consider $K \neq J$ w/ $K=1$ and $J=2$, so the l.h.s. $=0$ since $\delta_{12}=0$.

For the first version of the r.h.s. $A_{11}\,\mathrm{cof}(A_{21}) + A_{12}\,\mathrm{cof}(A_{22})$

$$a(-1)^{2+1}b + b(-1)^{2+2}a = -ab + ab = 0$$

while for the second version we have $A_{11}\,\mathrm{cof}(A_{12}) + A_{21}\,\mathrm{cof}(A_{22})$

$$a(-1)^{1+2}c + c(-1)^{2+2}a = -ac + ac = 0$$

This leads to a definition that will play a very important role:

The "classical adjoint" of an $n \times n$ matrix A is $\mathrm{adj}\,A$ w/ $(\mathrm{adj}\,A)_{ij} = \mathrm{cof}(A_{ji})$

Notice swapping of indices!!

If we take this and shove it into the previous theorem we have:

$$\delta_{KJ}\det A = \sum_j A_{Kj}\,\mathrm{cof}(A_{Jj}) = \sum_j A_{Kj}(\mathrm{adj}\,A)_{jJ} = (A\,\mathrm{adj}\,A)_{KJ}$$

and also

$$= \sum_j A_{jK}\,\mathrm{cof}(A_{jJ}) = \sum_j A_{jK}(\mathrm{adj}\,A)_{Jj} = [(\mathrm{adj}\,A)A]_{KJ}$$

Now this is a term by term equivalence, but since the indices are the same order, we can expand it to the entire matrix form: $I\det A = A\,\mathrm{adj}\,A = (\mathrm{adj}\,A)A$

This of course leads to:

The matrix has an inverse if and only if $\det A \neq 0$.

Proof:

if: Suppose $\det A \neq 0$, then we can divide by it getting $I = A\frac{\mathrm{adj}\,A}{\det A} = \frac{\mathrm{adj}\,A}{\det A}A \Rightarrow A^{-1} = \frac{\mathrm{adj}\,A}{\det A}$

only if: If there exists an A^{-1} then $AA^{-1} = I \Rightarrow \det(AA^{-1}) = 1 = \det A \det A^{-1} \Rightarrow \det A \neq 0$

So this is the matrix realization of the condition determining if A^{-1} exists which of course tells us the same for any lin. tran. isomorphic to A.

Consider a rotation acting on $\mathbb{R}^2$ which can be matricized by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

clearly $\det R_\theta = \cos^2\theta + \sin^2\theta = 1 \neq 0$ and so

$$R_\theta^{-1} = \frac{\operatorname{adj} R_\theta}{\det R_\theta} = \frac{1}{1} \underbrace{\begin{pmatrix} (-1)^{1+1}\cos\theta & (-1)^{1+2}\sin\theta \\ (-1)^{2+1}-\sin\theta & (-1)^{2+2}\cos\theta \end{pmatrix}}_{\operatorname{cof}(R_\theta)}^{T} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

so we need T to get $\operatorname{adj}(R_\theta)$

Sometimes, instead of calculating the determinant, we can use shortcuts.

Consider the operator D on P_n which we determined had no inverse since it wasn't 1-to-1 or onto. What about the matrix reflection?

Well recall that for P_1 D takes the form $D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in the $\{1, t\}$ basis, and $D = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$ in the $\{1+t, 1-t\}$ basis. From this, w/out taking the determinant formally we find:

From property 2 - in the $\{1, t\}$ basis clearly $\det D = 0 \Rightarrow D^{-1}$ does not exist.

From property 5 - the two rows of the $\{1+t, 1-t\}$ are the same, hence $\det D = 0 \Rightarrow$ no D^{-1}

From property 8 - since the two rows of the $\{1+t, 1-t\}$ are linearly dep. $\Rightarrow \det D = 0 \Rightarrow$ no D^{-1}

From not being lazy - taking the determinant in either basis $\Rightarrow \det D = 0 \Rightarrow$ no D^{-1}

And finally, if we consider a lin. tran. under a basis change $A \to A'$ we find:

$\det A = \det A'$