Mathematical Methods in Physics Midterm

1. Consider the set of elements {🙄, ☺, ☹} with composition * such that:

🙄 * ☺ = ☺ * 🙄 = ☺   ☹ * 🙄 = 🙄 * ☹ = ☹   🙄 * 🙄 = 🙄   ☹ * ☺ = ☺ * ☹ = 🙄   ☺ * ☺ = ☺   ☹ * ☹ = ☹

Do these form a group? If so, what is the multiplication table? If not, what group requirements are not met?

No, they do not form a group. While the first four compositions seem to imply that 🙄 is the identity, the last two seem to imply that ☺ and ☹ act as the identity as well. But a group must have only one identity!

2. For the vector space P_3 with an element $x = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3$ consider the linear transformation tD, that takes the derivative with respect to t then multiplies by t. Using the basis $\{1, t, t^2, t^3\}$ find the matrix version of this transformation. Then find the matrix form of $(tD)^n$.

Evaluating: $tDx = tD(\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3) = t(\alpha_1 + 2\alpha_2 t + 3\alpha_3 t^2) = (\alpha_1 t + 2\alpha_2 t^2 + 3\alpha_3 t^3)$

So as a matrix acting on the 4-tuple we have: $\begin{pmatrix} \qquad\qquad\qquad \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_1 \\ 2\alpha_2 \\ 3\alpha_3 \end{pmatrix}$ so by inspection we have $tD = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

And finally $(tD)^n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1^n & 0 & 0 \\ 0 & 0 & 2^n & 0 \\ 0 & 0 & 0 & 3^n \end{pmatrix}$.

3. Consider the matrix $M = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

a) Is this invertible? If so, what is M^{-1}?

Since $det \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = -5 \neq 0$, it is invertible with

$$M^{-1} = \frac{1}{detM} adjM = \frac{1}{-5}\begin{pmatrix} 1*3 & -1*2 \\ -1*4 & 1*1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{-1}{5} \end{pmatrix}.$$

b) Find the eigenvector(s). Are they linearly independent? Are they orthogonal?

For eigenvalues:

$\det(M-\lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda+1)(\lambda-5) = 0 \;\Rightarrow\; \lambda = -1{,}5$

Then for eigenvectors $Mv = \lambda v$:

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix} \;\Rightarrow\; \begin{matrix} a + 2b = -a \\ 4a + 3b = -b \end{matrix} \;\Rightarrow\; v_{-1} = \begin{pmatrix} a \\ -a \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 5\begin{pmatrix} a \\ b \end{pmatrix} \;\Rightarrow\; \begin{matrix} a + 2b = 5a \\ 4a + 3b = 5b \end{matrix} \;\Rightarrow\; v_5 = \begin{pmatrix} \frac{1}{2}b \\ b \end{pmatrix}$$

They are linearly independent $v_1 \neq k v_2$, but not orthogonal $(v_1, v_2) = \frac{1}{2}ab - ab \neq 0$.

c) Is this diagonalizable? If so what are its diagonal forms?

Yes it is!

To calculate its diagonal form we can start with $P = \begin{pmatrix} a & \frac{1}{2}b \\ -a & b \end{pmatrix}$ for which $P^{-1} = \frac{2}{3ab}\begin{pmatrix} b & -\frac{1}{2}b \\ a & a \end{pmatrix}$, then form $M_{diag} = P^{-1}MP = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$.

Of course we could have started with $P = \begin{pmatrix} \frac{1}{2}b & a \\ b & -a \end{pmatrix}$ for which $M_{diag} = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$.

4. Consider the vector space of polynomials up to 2nd degree with the usual inner product over the interval $t \in [-1,1]$. Starting with the obvious basis $X = \{x_0, x_1, x_2\} = \{1, t, t^2\}$, use Gram-Schmidt to determine an orthonormal basis (preferably and to your advantage starting with $x_0 = 1$).

First of all note that with the given interval, $(t^n, t^m) = \int_{-1}^{1} t^{n+m} dt = 0$ for $n + m =$ odd

We'll do $x_0 \to x_1 \to x_2$.

$$y_0 = \frac{x_0}{\|x_0\|} = \frac{1}{\sqrt{\int_{-1}^{1} dt}} = \frac{1}{\sqrt{2}}$$

$$y_1 = \frac{x_1 - (y_0, x_1) y_0}{\|x_1 - (y_0, x_1) y_0\|} = \frac{t - \left(\frac{1}{\sqrt{2}}, t\right)\frac{1}{\sqrt{2}}}{\left\|t - \left(\frac{1}{\sqrt{2}}, t\right)\frac{1}{\sqrt{2}}\right\|} = \frac{t}{\|t\|} = \frac{t}{\sqrt{\int_{-1}^{1} t^2 dt}} = \sqrt{\frac{3}{2}}\, t$$

$$y_2 = \frac{x_2 - (y_0, x_2) y_0 - (y_1, x_2) y_1}{\|x_2 - (y_0, x_2) y_0 - (y_1, x_2) y_1\|} = \frac{t^2 - \left(\frac{1}{\sqrt{2}}, t^2\right)\frac{1}{\sqrt{2}} - \left(\sqrt{\frac{3}{2}}t, t^2\right)\sqrt{\frac{3}{2}}t}{\left\|t^2 - \left(\frac{1}{\sqrt{2}}, t^2\right)\frac{1}{\sqrt{2}} - \left(\sqrt{\frac{3}{2}}t, t^2\right)\sqrt{\frac{3}{2}}t\right\|} = \frac{t^2 - \left(\frac{1}{\sqrt{2}}, t^2\right)\frac{1}{\sqrt{2}}}{\left\|t^2 - \left(\frac{1}{\sqrt{2}}, t^2\right)\frac{1}{\sqrt{2}}\right\|} = \frac{t^2 - \frac{1}{3}}{\sqrt{\int_{-1}^{1}\left(t^2 - \frac{1}{3}\right)^2 dt}} = \sqrt{\frac{45}{8}}\left(t^2 - \frac{1}{3}\right)$$

5. Consider the linear transformation acting in $\mathbb{R}^2$ which takes the matrix form $A_0 = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$. Now consider perturbing it with $\epsilon A_1 = \epsilon\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, to form $A = A_0 + \epsilon A_1 = \begin{pmatrix} 2 & -3+\epsilon \\ 1+\epsilon & -2 \end{pmatrix}$. Find the first and second order corrections to the eigenvalue(s).

First of all: $\det(A_0 - \lambda I) = \begin{vmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 3 = \lambda^2 - 1 = 0$

$\lambda_1 = 1 \quad \lambda_2 = -1$

Then: $A_0 v_1 = \lambda_1 v_1 \quad \Rightarrow \quad \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} 2a - 3b = a \\ a - 2b = b \end{matrix} \quad \Rightarrow \quad a = 3b \quad \Rightarrow v_1 = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix}$

And: $A_0 v_2 = \lambda_2 v_2 \quad \Rightarrow \quad \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} 2a - 3b = -a \\ a - 2b = -b \end{matrix} \quad \Rightarrow \quad a = b \quad \Rightarrow v_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Now we are ready. First:

$$\lambda_1^{(1)} = (v_1, A_1 v_1) = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix} = \frac{6}{10}$$

$$\lambda_2^{(1)} = (v_2, A_1 v_2) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 1$$

To get the second order eigenvalue correction we need the first order eigenvector correction.

$$v_1^{(1)} = \frac{(v_2, A v_1)}{\lambda_1 - \lambda_2} v_2 = \frac{\left(\frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{2}}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}}{2} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{2}{\sqrt{20}}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$v_2^{(1)} = \frac{(v_1, A v_2)}{\lambda_2 - \lambda_1} v_1 = \frac{\left(\frac{3}{\sqrt{10}} \ \frac{1}{\sqrt{10}}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}{-2} \begin{pmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix} = -\frac{2}{\sqrt{20}}\begin{pmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}$$

Then to get the second order corrections to the eigenvalues:

$$\lambda_1^{(2)} = \left(v_1, A v_1^{(1)}\right) = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{2}{\sqrt{20}}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{2}{5}$$

$$\lambda_2^{(2)} = \left(v_2, A v_2^{(1)}\right) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{-2}{\sqrt{20}}\begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix} = -\frac{2}{5}$$