1. a) For $SO(1,1)$ we can transform $dx^\mu = \begin{pmatrix} c\,dt \\ dx \end{pmatrix}$ with a boost $\Lambda^{\mu'}{}_{\mu} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$ where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta = \frac{v}{c}$

$\Lambda^{\mu'}{}_{\mu} \equiv \Lambda \Rightarrow \Lambda^{-1} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}$

To check that Λ is in $SO(1,1)$ first check that $\Lambda^T g \Lambda = g$

$$\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$$

$$\begin{pmatrix} -\gamma & -\beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$$

$$\begin{pmatrix} -\gamma^2+\beta^2\gamma^2 & \beta\gamma^2-\beta\gamma^2 \\ \beta\gamma^2-\beta\gamma^2 & -\beta^2\gamma^2+\gamma^2 \end{pmatrix} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \checkmark$$

> We will use the result below numerous times:
>
> $\gamma^2 - \beta^2\gamma^2 = \frac{1}{1-\beta^2} - \frac{\beta^2}{1-\beta^2} = \frac{1-\beta^2}{1-\beta^2} = 1$

Then check $\det\Lambda = +1$: $\det\Lambda = \gamma^2 - \beta^2\gamma^2 = +1$ ✓

b) $dx_\mu = g_{\mu\nu}dx^\nu = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}\begin{pmatrix} c\,dt \\ dx \end{pmatrix} = \begin{pmatrix} -c\,dt \\ dx \end{pmatrix}$ or $(-c\,dt \;\; dx)$

c) First: $dx_\mu dx^\mu = (-c\,dt \;\; dx)\begin{pmatrix} c\,dt \\ dx \end{pmatrix} = -c^2dt^2 + dx^2$

$$dx^\mu \to dx^{\mu'} = \Lambda^{\mu'}{}_{\mu}dx^\mu = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} c\,dt \\ dx \end{pmatrix} = \begin{pmatrix} \gamma c\,dt - \beta\gamma dx \\ -\beta\gamma c\,dt + \gamma dx \end{pmatrix}$$

$$dx_\mu \to dx_{\mu'} = \Lambda^{\mu}{}_{\mu'}dx_\mu = dx_\mu \left(\Lambda^{\mu'}{}_{\mu}\right)^{-1} = dx_\mu \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} = (-c\,dt \;\; dx)\begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}$$

$$dx_{\mu'} = (-\gamma c\,dt + \beta\gamma dx \quad -\beta\gamma c\,dt + \gamma dx)$$

$$dx_{\mu'}dx^{\mu'} = (-\gamma c\,dt + \beta\gamma dx \quad -\beta\gamma c\,dt + \gamma dx)\begin{pmatrix} \gamma c\,dt - \beta\gamma dx \\ -\beta\gamma c\,dt + \gamma dx \end{pmatrix}$$

$$= -\gamma^2c^2dt^2 - \beta^2\gamma^2dx^2 + \underbrace{\gamma^2\beta c\,dt\,dx + \gamma^2\beta c\,dt\,dx}_{} + \beta^2\gamma^2c^2dt^2 + \gamma^2dx^2 - \underbrace{\beta\gamma^2c\,dt\,dx - \beta\gamma^2c\,dt\,dx}_{}$$

Cancel

$$= -(\gamma^2 - \beta^2\gamma^2)c^2dt^2 + (\gamma^2 - \beta^2\gamma^2)dx^2$$

$$= -c^2dt^2 + dx^2 \checkmark$$

2. For $SO(2)$ we know $\Lambda^{\mu'}{}_{\mu} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ rotates $dx^{\mu} = \begin{pmatrix} dx \\ dy \end{pmatrix}$

Define $\Lambda \equiv \Lambda^{\mu'}{}_{\mu}$, then $\Lambda^{\mu}{}_{\mu'} = \Lambda^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \Lambda^{T} = \Lambda_{\mu}{}^{\mu'}$

and $\Lambda_{\mu'}{}^{\mu} = \Lambda^{-1\,T} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \Lambda$

So $\Lambda^{-1} = \Lambda^{T}$ and $\Lambda^{-1\,T} = \Lambda$ in this case, but is not true in general.

a) $M_{\mu\nu} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \to M_{\mu'\nu'} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} M_{\mu\nu} = \underbrace{\Lambda^{\mu}{}_{\mu'}}_{\text{need to transpose to get } \mu\text{'s adjacent}} M_{\mu\nu} \Lambda^{\nu}{}_{\nu'}$

$$= \Lambda^{-1\,T} M \Lambda^{-1} = \Lambda M \Lambda^{-1}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} A\cos\theta + C\sin\theta & B\cos\theta + D\sin\theta \\ A\sin\theta - C\cos\theta & B\sin\theta - D\cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} A\cos^2\theta + C\cos\theta\sin\theta + B\cos\theta\sin\theta + D\sin^2\theta & -A\cos\theta\sin\theta - C\sin^2\theta + B\cos^2\theta + D\cos\theta\sin\theta \\ A\cos\theta\sin\theta - C\cos^2\theta + B\sin^2\theta - D\cos\theta\sin\theta & -A\sin^2\theta + C\cos\theta\sin\theta + B\cos\theta\sin\theta - D\cos^2\theta \end{pmatrix}$$

b) $M^{\mu\nu} \to M^{\mu'\nu'} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} M^{\mu\nu} = \Lambda^{\mu'}{}_{\mu} M^{\mu\nu} \underbrace{\Lambda^{\nu'}{}_{\nu}}_{\text{transpose to get } \nu\text{'s adjacent}} = \Lambda M \Lambda^{T} = \Lambda M \Lambda^{-1}$ same as above!

c) $M_{\mu}{}^{\nu} \to M_{\mu'}{}^{\nu'} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu'}{}_{\nu} M_{\mu}{}^{\nu} = \underbrace{\Lambda^{\mu}{}_{\mu'}}_{\substack{\text{need to} \\ \text{transpose} \\ \text{to get} \\ \mu\text{'s adjacent}}} M_{\mu}{}^{\nu} \underbrace{\Lambda^{\nu'}{}_{\nu}}_{\substack{\text{transpose} \\ \text{to get} \\ \nu\text{'s adjacent}}} = (\Lambda^{-1})^{T} M \Lambda^{T} = \Lambda M \Lambda^{-1}$ same as above!

d) $M^{\mu}{}_{\nu} \to M^{\mu'}{}_{\nu'} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu}{}_{\nu'} M^{\mu}{}_{\nu} = \Lambda^{\mu'}{}_{\mu} M^{\mu}{}_{\nu} \Lambda^{\nu}{}_{\nu'} = \Lambda M \Lambda^{-1}$ same as above!

So oddly it seems that in this case all of $M^{\mu\nu}$, $M_{\mu\nu}$, $M^{\mu}{}_{\nu}$ and $M_{\mu}{}^{\nu}$ transform the same. To understand why, recall that for $SO(2)$, our metric is $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which means that vectors and dual vectors have the same components, i.e. there is really no reason to distinguish them. But that is what upper and lower indices indicate. So no difference between vectors & dual vectors $\Rightarrow$ no difference between upper and lower indices, hence $M^{\mu\nu} \sim M_{\mu\nu} \sim M^{\mu}{}_{\nu} \sim M_{\mu}{}^{\nu}$.

Note: This is not true in general!!

3. a) $dx_\mu dx^\mu$ $(0,0)$-tensor $\quad dx_\mu dx^\mu \to dx_{\mu'} dx^{\mu'} = dx_\mu dx^\mu$

b) $dx_\mu dx^\nu$ $(1,1)$-tensor $\quad dx_\mu dx^\nu \to dx_{\mu'} dx^{\nu'} = \Lambda_{\mu'}{}^{\mu} \Lambda^{\nu'}{}_{\nu}\, dx_\mu dx^\nu$

c) $dx_\mu dx_\nu$ $(0,2)$-tensor $\quad dx_\mu dx^\nu \to dx_{\mu'} dx_{\nu'} = \Lambda_{\mu'}{}^{\mu} \Lambda_{\nu'}{}^{\nu}\, dx_\mu dx_\nu$

d) $T^{\mu\nu} W^{\rho}_{\mu\lambda\kappa} = M^{\nu\rho}{}_{\lambda\kappa}$ $(2,2)$-tensor $\quad M^{\nu\rho}{}_{\lambda\kappa} \to M^{\nu'\rho'}{}_{\lambda'\kappa'} = \Lambda^{\nu'}{}_{\nu} \Lambda^{\rho'}{}_{\rho} \Lambda_{\lambda'}{}^{\lambda} \Lambda_{\kappa'}{}^{\kappa} M^{\nu\rho}{}_{\lambda\kappa}$

4. a) $A + B \rightarrow C + D + E + F$

Since B begins at rest $\vec{p}_B = 0$. We want to find (in terms of $m_A, m_B, m_C, m_D, m_E, m_F$) the minimum E_A needed for this process to occur.

Important: In any process in which new particles are created, then in the center of momentum frame (where $\vec{p}_{tot} = 0$) the minimum total energy the new particles must have is generally their rest energy. If more than enough total energy is present, the final particles will also be moving, but again in the minimum energy case they are created at rest in the C.O.M. frame.

So in the Lab frame we have: $P^\mu_{tot,lab} = P_A^\mu + P_B^\mu = \begin{pmatrix} E_A/c \\ \vec{p}_A \end{pmatrix} + \begin{pmatrix} m_B c \\ \vec{0} \end{pmatrix} = \begin{pmatrix} E_A/c + m_B c \\ \vec{p}_A \end{pmatrix}$

Meanwhile in the C.O.M. frame: $P^\mu_{tot,com} = P_C^\mu + P_D^\mu + P_E^\mu + P_F^\mu = \begin{pmatrix} m_C c \\ \vec{0} \end{pmatrix} + \begin{pmatrix} m_D c \\ \vec{0} \end{pmatrix} + \begin{pmatrix} m_E c \\ \vec{0} \end{pmatrix} + \begin{pmatrix} m_F c \\ \vec{0} \end{pmatrix}$

Note: The minimum energy required leaves the outgoing particles at rest. $= \begin{pmatrix} m_C c + m_D c + m_E c + m_F c \\ \vec{0} \end{pmatrix}$

Now we cannot say $P^\mu_{tot,lab} = P^\mu_{tot,com}$ but we can say $\underbrace{(P_{tot\,\mu} P^\mu_{tot})_{lab} = (P_{tot\,\mu} P^\mu_{tot})_{com}}_{\text{Since these are invariants, i.e. the same in } \underline{\text{any}} \text{ frame.}}$

Then: $(P_{A\mu} + P_{B\mu})(P_A^\mu + P_B^\mu)_{lab} = \left(-\frac{E_A}{c} - m_B c \quad \vec{p}_A\right)\begin{pmatrix} E_A/c + m_B c \\ \vec{p}_A \end{pmatrix} = -\frac{E_A^2}{c^2} - m_B^2 c^2 - 2E_A m_B + p_A^2$

$= -m_A^2 c^2 - m_B^2 c^2 - 2E_A m_B$ Recall: $\frac{E_A^2}{c^2} - p_A^2 = m_A^2 c^2$ always!

And: $(P_{C\mu} + P_{D\mu} + P_{E\mu} + P_{F\mu})(P_C^\mu + P_D^\mu + P_E^\mu + P_F^\mu)_{com} = -(m_C c + m_D c + m_E c + m_F c)^2$

Finally: $-m_A^2 c^2 - m_B^2 c^2 - 2E_A m_B = -(m_C c + m_D c + m_E c + m_F c)^2$

$$E_A = \frac{1}{2m_B}\left[\underbrace{(m_C c + m_D c + m_E c + m_F c)^2}_{\text{This can be generalized to any number of final particles.}} - m_A^2 c^2 - m_B^2 c^2\right]$$