1. $\Gamma_1 = \frac{S}{2\hbar m_1} \int |M|^2 (2\pi)^4 \delta^4(p_1 - p_2 - \cdots - p_n) \prod_{j=2}^{n} 2\pi\, \delta(p_j^2 - m_j^2 c^2)\, \Theta(p_j^0) \frac{d^4 p_j}{(2\pi)^4}$

$= \frac{S}{2\hbar m_1} \int |M|^2 (2\pi)^4 \delta^4(p_1 - p_2 - \cdots - p_n) \prod_{j=2}^{n} \frac{1}{2\sqrt{\vec{p}_j^{\,2} + m_j^2 c^2}} \frac{d^3 \vec{p}_j}{(2\pi)^3}$

In rest frame of particle 1 we have $p_1 = \begin{pmatrix} m_1 c \\ \vec{0} \end{pmatrix}$ while $p_2 = \begin{pmatrix} m_2 \gamma_2 c \\ m_2 \gamma_2 \vec{v}_2 \end{pmatrix}$, $p_3 = \begin{pmatrix} m_3 \gamma_3 c \\ m_3 \gamma_3 \vec{v}_3 \end{pmatrix}$, etc.

In order for $\delta^4(p_1 - p_2 - \cdots - p_n)$ to give a nonzero contribution we need $p_1^\mu - p_2^\mu - \cdots - p_n^\mu = 0^\mu$

But for the timelike terms this requires: $m_1 c - m_2 \gamma_2 c - \cdots - m_n \gamma_n c = 0$

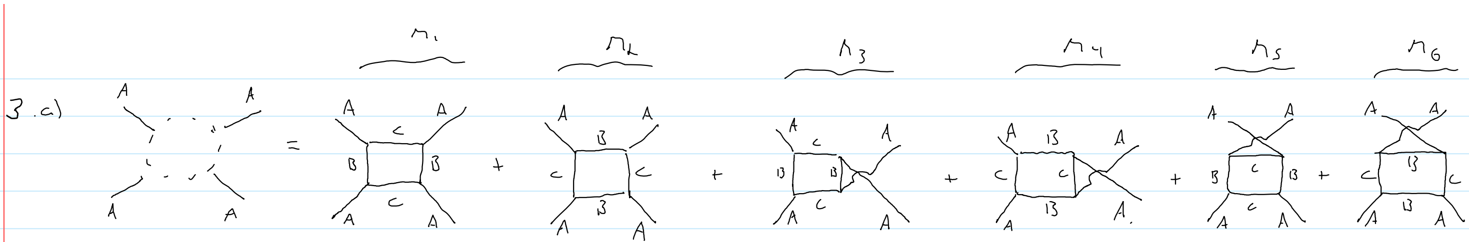
$m_1 - m_2 \gamma_2 - \cdots - m_n \gamma_n = 0 \Rightarrow m_1 = m_2 \gamma_2 + \cdots + m_n \gamma_n$

However all $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \geq 1$ so if $m_2 + \cdots + m_n > m_1$ then this can never be satisfied.

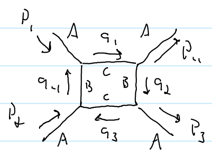
2. For decays, we can go to the rest frame of the decaying particle in which the only energy is mc^2. For a scattering event, we need relative motion for the colliding particles, so in no frame is the incoming energy only equal to the masses of the incoming particles ($\times c^2$). But this nonzero "kinetic energy" can be converted to the mass of the exiting particles, thereby raising the total outgoing mass to a higher value than the incoming.

If the incoming and outgoing masses are the same, we still get a variety of possible outcomes based on the direction in which the final particles will travel.

3.c)



b)



$$\iiiint (-ig)^4 \frac{i}{q_1^2-m_C^2c^2}\frac{i}{q_2^2-m_B^2c^2}\frac{i}{q_3^2-m_C^2c^2}\frac{i}{q_4^2-m_B^2c^2}(2\pi)^4\delta^4(p_1+q_4-q_1)(2\pi)^4\delta^4(q_1-p_4-q_2)$$
$$(2\pi)^4\delta^4(q_2-p_3-q_3)(2\pi)^4\delta^4(p_2+q_3-q_4)\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}\frac{d^4q_4}{(2\pi)^4}$$

Use $\delta^4(p_1+q_4-q_1) \Rightarrow -p_1+q_1=q_4$ to eliminate (or integrate over) q_4

$$\iiint (-ig)^4 \frac{i}{q_1^2-m_C^2c^2}\frac{i}{q_2^2-m_B^2c^2}\frac{i}{q_3^2-m_C^2c^2}\frac{i}{(-p_1+q_1)^2-m_B^2c^2}(2\pi)^4\delta^4(q_1-p_4-q_2)\,(2\pi)^4\delta^4(q_2-p_3-q_3)$$
$$(2\pi)^4\delta^4(p_2+q_3+p_1-q_1)\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}$$

Use $\delta^4(q_1-p_4-q_2) \Rightarrow q_1=p_4+q_2$ to integrate over q_1

$$\iint (-ig)^4 \frac{i}{(p_4+q_2)^2-m_C^2c^2}\frac{i}{q_2^2-m_B^2c^2}\frac{i}{q_3^2-m_C^2c^2}\frac{i}{(-p_1+p_4+q_2)^2-m_B^2c^2}(2\pi)^4\delta^4(q_2-p_3-q_3)(2\pi)^4\delta^4(p_2+q_3+p_1-p_4-q_2)$$
$$\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}$$

use $\delta^4(q_2-p_3-q_3) \Rightarrow q_3=q_2-p_3$ to integrate over q_3 and let $q_2=q$

$$\int (-ig)^4 \frac{i}{(p_4+q)^2-m_C^2c^2}\frac{i}{q^2-m_B^2c^2}\frac{i}{(q-p_3)^2-m_C^2c^2}\frac{i}{(-p_1+p_4+q)^2-m_B^2c^2}(2\pi)^4\delta^4(p_2+\cancel{q}-p_3+p_1-p_4-\cancel{q})\frac{d^4q}{(2\pi)^4}$$

$$\mathcal{M}_1 = i\int (-ig)^4 \frac{i}{(p_4+q)^2-m_C^2c^2}\frac{i}{q^2-m_B^2c^2}\frac{i}{(q-p_3)^2-m_C^2c^2}\frac{i}{(-p_1+p_4+q)^2-m_B^2c^2}\frac{d^4q}{(2\pi)^4}$$

Now for $\mathcal{M}_2$ we simply interchange $m_C \leftrightarrow m_B$ everywhere in $\mathcal{M}_1$

$$\mathcal{M}_2 = i\int (-ig)^4 \frac{i}{(p_4+q)^2-m_B^2c^2}\frac{i}{q^2-m_C^2c^2}\frac{i}{(q-p_3)^2-m_B^2c^2}\frac{i}{(-p_1+p_4+q)^2-m_C^2c^2}\frac{d^4q}{(2\pi)^4}$$

To get $\mathcal{M}_3$ we simply interchange $p_3 \leftrightarrow p_4$ everywhere in $\mathcal{M}_1$

$$\mathcal{M}_3 = i\int (-ig)^4 \frac{i}{(p_3+q)^2-m_C^2c^2}\frac{i}{q^2-m_B^2c^2}\frac{i}{(q-p_4)^2-m_C^2c^2}\frac{i}{(-p_1+p_3+q)^2-m_B^2c^2}\frac{d^4q}{(2\pi)^4}$$

And to get $\mathcal{M}_4$ we can interchange $m_C \leftrightarrow m_B$ in $\mathcal{M}_3$

$$\mathcal{M}_4 = i\int (-ig)^4 \frac{i}{(p_3+q)^2-m_B^2c^2}\frac{i}{q^2-m_C^2c^2}\frac{i}{(q-p_4)^2-m_B^2c^2}\frac{i}{(-p_1+p_3+q)^2-m_C^2c^2}\frac{d^4q}{(2\pi)^4}$$

or we could have swapped $p_3 \leftrightarrow p_4$ in $\mathcal{M}_2$

$$\mathcal{M}_4 = i\int (-ig)^4 \frac{i}{(p_3+q)^2-m_B^2c^2}\frac{i}{q^2-m_C^2c^2}\frac{i}{(q-p_4)^2-m_B^2c^2}\frac{i}{(-p_1+p_3+q)^2-m_C^2c^2}\frac{d^4q}{(2\pi)^4}$$

To get $\mathcal{M}_5$ we interchange $p_1 \leftrightarrow p_4$ in $\mathcal{M}_1$

$$\mathcal{M}_5 = i\int (-ig)^4 \frac{i}{(p_1+q)^2-m_C^2c^2}\frac{i}{q^2-m_B^2c^2}\frac{i}{(q-p_3)^2-m_C^2c^2}\frac{i}{(-p_4+p_1+q)^2-m_C^2c^2}$$

And to get $\mathcal{M}_6$ we interchange $p_1 \rightarrow p_4$ in $\mathcal{M}_2$

$$\mathcal{M}_6 = i \int (-ig) \quad (p_2+q)^2 - m_B^2c^2 \quad q^2 - m_c^2 c \quad (q-p_3)^2 - m_B^2c^2 \quad (-p_4+p_1+q)^2 - m_c^2c^2$$

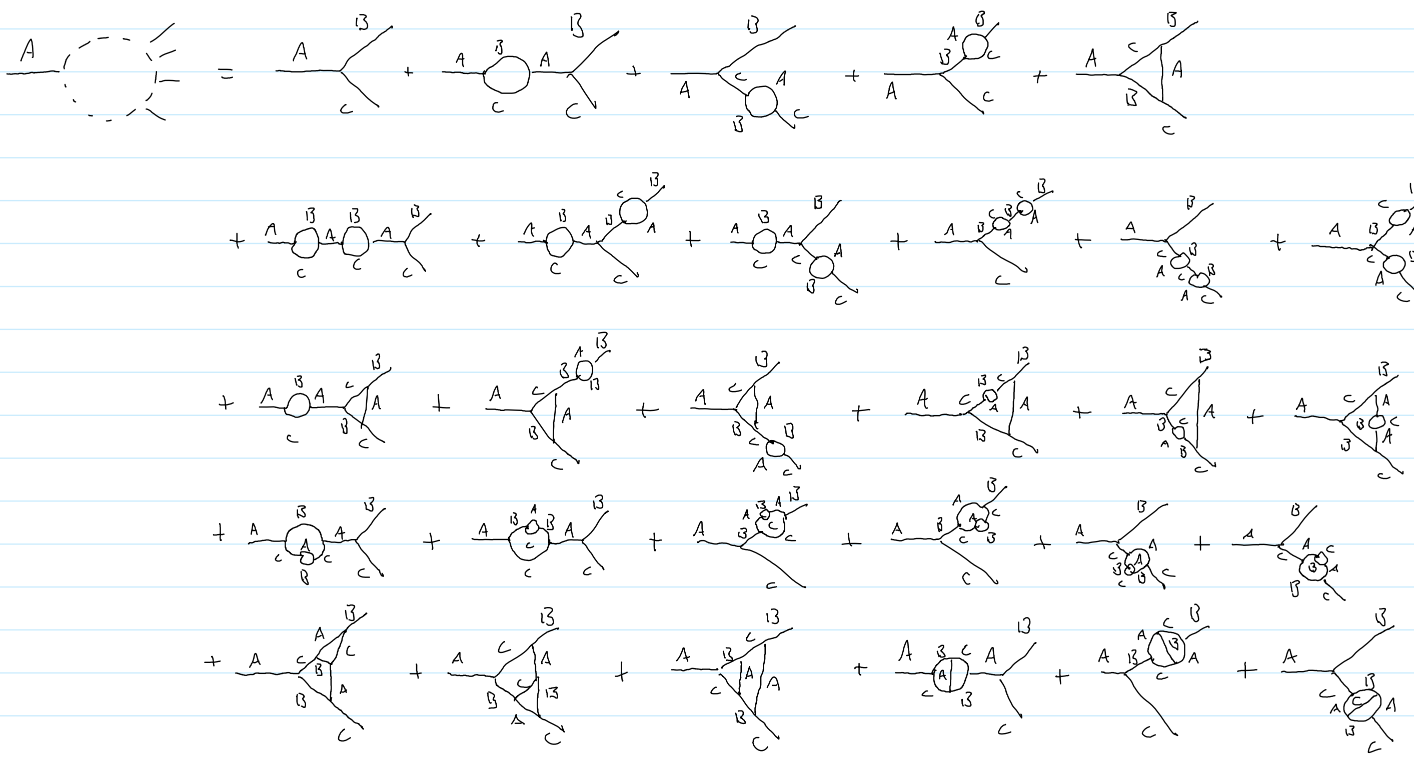
You can now set $m_B = m_C = 0$ in all of these! Then $\underbrace{\mathcal{M}_1 = \mathcal{M}_2}_{\mathcal{M}_{12}}$ and $\underbrace{\mathcal{M}_3 = \mathcal{M}_4}_{\mathcal{M}_{34}}$ and $\underbrace{\mathcal{M}_5 = \mathcal{M}_6}_{\mathcal{M}_{56}}$, so we get identical contributions.

$$\mathcal{M}_{12} = i \int (-ig)^4 \frac{i}{(p_4+q)^2} \frac{i}{q^2} \frac{i}{(q-p_3)^2} \frac{i}{(-p_1+p_4+q)^2} \frac{d^4q}{(2\pi)^4} \qquad \mathcal{M}_{56} = i \int (-ig)^4 \frac{i}{(p_2+q)^2} \frac{i}{q^2} \frac{i}{(q-p_3)^2} \frac{i}{(-p_4+p_1+q)^2} \frac{d^4q}{(2\pi)^4}$$

$$\mathcal{M}_{34} = i \int (-ig)^4 \frac{i}{(p_3+q)^2} \frac{i}{q^2} \frac{i}{(q-p_4)^2} \frac{i}{(-p_1+p_3+q)^2} \frac{d^4q}{(2\pi)^4}$$

c) $$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|}{(E_1+E_2)^2} \frac{|\vec{p}_f|}{|\vec{p}_i|} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{\underbrace{|2\mathcal{M}_{12}+2\mathcal{M}_{34}+2\mathcal{M}_{56}|}}{2(E_1+E_2)^2} \frac{|\vec{p}_f|}{|\vec{p}_i|}$$

4.



5. $\bar{u}^{(1)} u^{(1)} = u^{(1)\dagger} \gamma^0 u^{(1)}$

$$= \frac{E+mc^2}{c}\begin{pmatrix} 1 & 0 & \frac{cp_z}{E+mc^2} & \frac{c(p_x-ip_y)}{E+mc^2}\end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1\end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2}\end{pmatrix}$$

$$= \frac{E+mc^2}{c}\begin{pmatrix} 1 & 0 & \frac{cp_z}{E+mc^2} & \frac{c(p_x-ip_y)}{E+mc^2}\end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -\frac{cp_z}{E+mc^2} \\ -\frac{c(p_x+ip_y)}{E+mc^2}\end{pmatrix}$$

$$= \frac{E+mc^2}{c}\left(1 + 0 - \frac{c^2 p_z^2}{(E+mc^2)^2} - \frac{c^2(p_x^2+p_y^2)}{(E+mc^2)^2}\right) = \frac{E+mc^2}{c}\left(1 - \frac{c^2p^2}{(E+mc^2)^2}\right)$$

Using that $E^2 - p^2c^2 = m^2c^4 \Rightarrow p^2c^2 = E^2 - m^2c^4 \Rightarrow 1 - \frac{c^2p^2}{(E+mc^2)^2} = \frac{(E+mc^2)^2 - E^2 + m^2c^4}{(E+mc^2)^2} = \frac{2Emc^2 + 2m^2c^4}{(E+mc^2)^2}$

$$\bar{u}^{(1)} u^{(1)} = \frac{E+mc^2}{c}\,\frac{2mc^2(E+mc^2)}{(E+mc^2)^2} = 2mc \quad \checkmark$$

$\bar{u}^{(2)} u^{(1)} = u^{(2)\dagger}\gamma^0 u^{(1)}$ and using $\gamma^0 u^{(1)}$ from above

$$= \frac{E+mc^2}{c}\begin{pmatrix} 0 & 1 & \frac{c(p_x+ip_y)}{E+mc^2} & \frac{-cp_z}{E+mc^2}\end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -\frac{cp_z}{E+mc^2} \\ -\frac{c(p_x+ip_y)}{E+mc^2}\end{pmatrix}$$

$$= \frac{E+mc^2}{c}\left(0 + 0 - \frac{c^2 p_z(p_x+ip_y)}{E+mc^2} + \frac{c^2 p_z(p_x+ip_y)}{E+mc^2}\right) = 0 \quad \checkmark$$

6. $\sum_s u^{(s)}\bar{u}^{(s)} = u^{(1)}\bar{u}^{(1)} + u^{(2)}\bar{u}^{(2)}$

$$= u^{(1)}u^{(1)\dagger}\gamma^0 + u^{(2)}u^{(2)\dagger}\gamma^0$$

$$= \frac{E+mc^2}{c}\begin{pmatrix}1\\0\\ \frac{cp_z}{E+mc^2}\\ \frac{c(p_x+ip_y)}{E+mc^2}\end{pmatrix}\begin{pmatrix}1 & 0 & \frac{cp_z}{E+mc^2} & \frac{c(p_x-ip_y)}{E+mc^2}\end{pmatrix}\begin{pmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{pmatrix}$$

$$+ \frac{E+mc^2}{c}\begin{pmatrix}0\\1\\ \frac{c(p_x-ip_y)}{E+mc^2}\\ -\frac{cp_z}{E+mc^2}\end{pmatrix}\begin{pmatrix}0 & 1 & \frac{c(p_x+ip_y)}{E+mc^2} & \frac{-cp_z}{E+mc^2}\end{pmatrix}\begin{pmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{pmatrix}$$

$$= \frac{E+mc^2}{c}\begin{pmatrix}1\\0\\ \frac{cp_z}{E+mc^2}\\ \frac{c(p_x+ip_y)}{E+mc^2}\end{pmatrix}\begin{pmatrix}1 & 0 & \frac{-cp_z}{E+mc^2} & \frac{-c(p_x-ip_y)}{E+mc^2}\end{pmatrix}$$

$$+ \frac{E+mc^2}{c}\begin{pmatrix}0\\1\\ \frac{c(p_x-ip_y)}{E+mc^2}\\ -\frac{cp_z}{E+mc^2}\end{pmatrix}\begin{pmatrix}0 & 1 & \frac{-c(p_x+ip_y)}{E+mc^2} & \frac{+cp_z}{E+mc^2}\end{pmatrix}$$

Use $\begin{pmatrix}a\\b\end{pmatrix}\begin{pmatrix}c & d\end{pmatrix} = \begin{pmatrix}ac & ad\\ bc & bd\end{pmatrix}$

$$= \frac{E+mc^2}{c}\begin{pmatrix}1 & 0 & \frac{-cp_z}{E+mc^2} & -\frac{c(p_x-ip_y)}{E+mc^2}\\ 0 & 0 & 0 & 0\\ \frac{cp_z}{E+mc^2} & 0 & \frac{-c^2p_z^2}{(E+mc^2)^2} & \frac{-c^2p_z(p_x-ip_y)}{(E+mc^2)^2}\\ \frac{c(p_x+ip_y)}{E+mc^2} & 0 & \frac{-c^2p_z(p_x+ip_y)}{E+mc^2} & \frac{-c^2(p_x^2+p_y^2)}{E+mc^2}\end{pmatrix}$$

$$+ \frac{E+mc^2}{c}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1 & \frac{-c(p_x+ip_y)}{E+mc^2} & \frac{cp_z}{E+mc^2}\\ 0 & \frac{c(p_x-ip_y)}{E+mc^2} & \frac{-c^2(p_x^2+p_y^2)}{(E+mc^2)^2} & \frac{c^2p_z(p_x-ip_y)}{(E+mc^2)^2}\\ 0 & \frac{-cp_z}{E+mc^2} & \frac{c^2p_z(p_x+ip_y)}{(E+mc^2)^2} & \frac{-c^2p_z^2}{(E+mc^2)^2}\end{pmatrix}$$

$$= \frac{E+mc^2}{c}\begin{pmatrix}1 & 0 & \frac{-cp_z}{E+mc^2} & \frac{-c(p_x-ip_y)}{E+mc^2}\\ 0 & 1 & \frac{-c(p_x+ip_y)}{E+mc^2} & \frac{cp_z}{E+mc^2}\\ \frac{cp_z}{E+mc^2} & \frac{c(p_x-ip_y)}{E+mc^2} & \frac{-c^2p^2}{(E+mc^2)^2} & 0\\ \frac{c(p_x+ip_y)}{E+mc^2} & \frac{-cp_z}{E+mc^2} & 0 & \frac{-c^2p^2}{(E+mc^2)^2}\end{pmatrix} = \begin{pmatrix}\frac{E}{c}+mc & 0 & -p_z & -p_x+ip_y\\ 0 & \frac{E}{c}+mc & -p_x-ip_y & p_z\\ p_z & p_x-ip_y & -\frac{E}{c}+mc & 0\\ p_x+ip_y & -p_z & 0 & -\frac{E}{c}+mc\end{pmatrix}$$

Since $E^2 - p^2c^2 = m^2c^4 \Rightarrow p^2c^2 = E^2 - m^2c^4 = (E-mc^2)(E+mc^2)$

$$\frac{-(E-mc^2)}{(E+mc^2)^2} = \frac{-(E-mc^2)}{(E+mc^2)}$$

$$\gamma^\mu p_\mu + mc = \gamma^0 p^0 - \gamma^1 p^1 - \gamma^2 p^2 - \gamma^3 p^3 + mc = \gamma^0 \frac{E}{c} - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z + mc$$

$$= \begin{pmatrix} \frac{E}{c} & & & \\ & \frac{E}{c} & & \\ & & -\frac{E}{c} & \\ & & & -\frac{E}{c} \end{pmatrix} - \begin{pmatrix} & & & p_x \\ & & p_x & \\ & -p_x & & \\ -p_x & & & \end{pmatrix} - \begin{pmatrix} & & & -ip_y \\ & & ip_y & \\ & ip_y & & \\ -ip_y & & & \end{pmatrix} - \begin{pmatrix} & & p_z & \\ & & & -p_z \\ -p_z & & & \\ & p_z & & \end{pmatrix} + \begin{pmatrix} mc & & & \\ & mc & & \\ & & mc & \\ & & & mc \end{pmatrix}$$

$$= \begin{pmatrix} \frac{E}{c}+mc & 0 & -p_z & -p_x+ip_y \\ 0 & \frac{E}{c}+mc & -p_x-ip_y & p_z \\ p_z & p_x-ip_y & -\frac{E}{c}+mc & 0 \\ p_x+ip_y & -p_z & 0 & -\frac{E}{c}+mc \end{pmatrix}$$

Which agrees w/ $\sum_s u^{(s)} \bar{u}^{(s)}$

7. To compute the inverse of $\gamma^\mu p_\mu - mc$ we can steal a result from HW4 # 5 that:

$$(\gamma^\mu p_\mu - mc)(\gamma^\nu p_\nu + mc) = \gamma^\mu\gamma^\nu p_\mu p_\nu - m^2c^2 = \frac{1}{2}\gamma^\mu\gamma^\nu \underbrace{p_\mu p_\nu + \frac{1}{2}\gamma^\mu\gamma^\nu p_\nu p_\mu}_{\text{can be freely switched}} - m^2c^2$$

Then relabelling $\mu \leftrightarrow \nu$ in second term

$$= \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)p_\mu p_\nu - m^2c^2$$

$$= \eta^{\mu\nu} p_\mu p_\nu - m^2c^2 = p^2 - m^2c^2$$

So we have $\underbrace{(\gamma^\mu p_\mu - mc)}_{M_1}\underbrace{(\gamma^\nu p_\nu + mc)}_{M_2} = (p^2 - m^2c^2)\underbrace{I}_{M_3} \Rightarrow \underbrace{(\gamma^\mu p_\mu - mc)}_{M}\underbrace{\frac{(\gamma^\nu p_\nu + mc)}{p^2 - m^2c^2}}_{M^{-1}} = I$

Then $iM^{-1} = \dfrac{i(\gamma^\mu p_\mu + mc)}{p^2 - m^2c^2}$