

Consider the 3 equations of motion we have discussed so far:

Spin 0: $\partial_\mu \partial^\mu \phi - \left(\frac{\hbar c}{\hbar}\right)^2 \phi = 0$ K-G equation

Spin 1/2: $\not{\partial} \psi + \frac{\hbar c}{\hbar} \psi = 0 \Rightarrow$ You showed in your HW that $\Rightarrow \partial_\mu \partial^\mu \psi - \left(\frac{\hbar c}{\hbar}\right)^2 \psi = 0$

Spin 1: $\partial_\mu F^{\mu\nu} - \left(\frac{\hbar c}{\hbar}\right)^2 A^\nu = 0 \Rightarrow \underbrace{\partial_\nu \partial_\mu F^{\mu\nu}}_{=0 \text{ since sym-antisym}} - \left(\frac{\hbar c}{\hbar}\right)^2 \underbrace{\partial_\nu A^\nu}_{=0} = 0$

K-G equation
holds for each component of ψ and A^ν separately!

$\Rightarrow \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) - \left(\frac{\hbar c}{\hbar}\right)^2 A^\nu = \underbrace{\partial_\mu \partial^\mu A^\nu - \left(\frac{\hbar c}{\hbar}\right)^2 A^\nu}_{\text{K-G equation}} = 0$

Why does everything also satisfy the Klein-Gordon equation?

Consider: $\frac{E^2}{c^2} - p^2 = m^2 c^2 \Rightarrow p_\mu p^\mu + m^2 c^2 = 0$ let $p_\mu = i\hbar \partial_\mu$ and act on ϕ

$- \hbar^2 \partial_\mu \partial^\mu \phi + m^2 c^2 \phi = 0 \Rightarrow \underbrace{\partial_\mu \partial^\mu \phi - \left(\frac{\hbar c}{\hbar}\right)^2 \phi}_{\text{K-G equation}} = 0$

Note: Starting w/ $\frac{p^2}{\hbar^2} + V = E$
 using $p \rightarrow -i\hbar \vec{\nabla}$, $E \rightarrow i\hbar \frac{\partial}{\partial t}$
 then $-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi = i\hbar \frac{\partial \psi}{\partial t}$
 Non-relativistic Time-dependent Schrodinger Equation

K-G equation is really just reflecting the mass-shell condition which all real degrees of freedom must satisfy!

So why do we have Dirac and Proca? What more do they tell us?

To understand this we need to think about degrees of freedom. Typically the # d.o.f. is equal to the dimension of configuration space or half the dimension of phase space.

Example: 1 non-relativistic spin 0 particle is described by: $x(t), y(t), z(t) \Rightarrow$ 3D configuration space
or: $x(t), y(t), z(t), v_x(t), v_y(t), v_z(t)$ 6D phase space
 \Rightarrow 3 d.o.f. ($3N$ for N particles)

For fields the counting is more subtle. Technically a field has an ∞ # of d.o.f. since we must specify the field value at each spacetime point. However we know that the "motion" of particle-like field fluctuations is covered by K-G equation, so we can set that aside and ask how many d.o.f. per point are left over. For spin 0 scalars the K-G equation does it all.

But what if a particle has nontrivial internal spin? Long ago in a land far away, Eugene Wigner developed a method for identifying the degrees of freedom using some of the deeper elements of group theory (see "Wigner's classification" or "induced representations"). I will try to give you a sense of the result.

The main idea (in physics) is that the "motion" of a particle, captured by the 4-momentum P^μ , partly characterizes the freedom in specifying what a particle is doing (its d.o.f.s). The question then is "after specifying P^μ , what else can be chosen without changing P^μ ."

The good news is that counting d.o.f. is independent of reference frame (as it should be in relativity) so we can use the simplest one.

Consider a particle in its rest frame: $P_{\text{rest}}^\mu = \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix}$ } The transformations which leave this invariant are clearly 3D rotations leaving $\vec{p} = \vec{0}$ unchanged.

It is not intuitive, but we have just as much freedom left over in any other frame.

The upshot is that to characterize the internal spin states, we get to carry over all of the familiar angular momentum information from 3D NR QM.

So let's review some essential results for angular momentum in 3D NR QM.

First of all, in contrast to linear momentum, angular momentum is always discrete in QM. recall this is continuous for ∞ dimension and discrete for finite dimensions

The reason is that the angle through which we rotate is compact, i.e. $\theta \in (0, 2\pi]$. (like particle in a box)

For quantized internal spin, the total spin magnitude S^2 is fixed once and for all. But we can dynamically change S_z . This is not true for orbital angular momentum where L^2 and L_z can change.

From QM you know that if: $S \sim 0 \Rightarrow S_z = 0$ 1 d.o.f.
 $S \sim \frac{1}{2} \Rightarrow S_z = \pm \frac{1}{2} \hbar$ 2 d.o.f.
 $S \sim 1 \Rightarrow S_z = (-1, 0, +1)\hbar$ 3 d.o.f.
etc.

(actually, $S^2 = s(s+1)\hbar^2$)
Note: $S^2 > S_z^2 = s^2 \hbar^2$

We can never have $S = S_z$ since then $S_x = S_y = 0$, but we know these 3 cannot be simultaneously known.

As an example consider A^ν and the Proca equation. Naively, A^ν should have 4 real parameters to be specified, but being spin 1, we should really only expect 3 (+1, 0, -1).

$$\partial_\mu F^{\mu\nu} - \left(\frac{\hbar c}{\lambda}\right)^2 A^\nu = 0$$

Go to momentum space via $i\hbar\partial_\mu \rightarrow p_\mu$ ($\partial_\mu \rightarrow -\frac{i}{\hbar}p_\mu$) then:

$$-\frac{i}{\hbar}p_\mu \left(-\frac{i}{\hbar}p^\mu A^\nu + \frac{i}{\hbar}p^\nu A^\mu\right) - \left(\frac{\hbar c}{\lambda}\right)^2 A^\nu = 0 = -\frac{1}{\hbar^2}p_\mu (p^\mu A^\nu - p^\nu A^\mu) - \left(\frac{\hbar c}{\lambda}\right)^2 A^\nu$$

In the rest frame $p_\mu = (-\hbar c, \vec{0})$ so we have:

$$\begin{aligned} v=0 & \quad \frac{1}{\hbar^2}(\hbar c)^2 A^0 + \frac{1}{\hbar^2}(\hbar c)^2 A^0 - \left(\frac{\hbar c}{\lambda}\right)^2 A^0 = 0 \Rightarrow A^0 = 0 \\ v \neq 0 & \quad \frac{1}{\hbar^2}(\hbar c)^2 A^i - \left(\frac{\hbar c}{\lambda}\right)^2 A^i = 0 \end{aligned} \quad \left. \vphantom{\begin{aligned} v=0 \\ v \neq 0 \end{aligned}} \right\} \text{So we have 3 values of } A^i \text{ to select}$$

As for spinors ψ and the Dirac equation, we know that naively ψ has 4 complex components or 8 real degrees of freedom. But a spin $\frac{1}{2}$ particle should only have 2 ($\pm \frac{1}{2}$) so the Dirac equation must do a lot of cutting down.

To see this consider the Dirac equation: $\gamma^\mu \partial_\mu \psi + \frac{\hbar c}{\lambda} \psi = 0$ in momentum representation ($i\hbar\partial_\mu \rightarrow p_\mu$)
 $-i\gamma^\mu p_\mu \psi + \hbar c \psi = 0$

In the rest frame $p_\mu = (-\hbar c, 0, 0, 0)$ so this becomes: $i\gamma^0 \hbar c \psi + \hbar c \psi = 0$
 $(i\gamma^0 + 1)\hbar c \psi = 0$

$$\text{Recall: } \gamma^0 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow i\gamma^0 + 1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{Then: } (i\gamma^0 + 1)\hbar c \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \hbar c \begin{pmatrix} \psi_1 + \psi_3 \\ \psi_2 + \psi_4 \\ \psi_1 + \psi_3 \\ \psi_2 + \psi_4 \end{pmatrix} = 0 \Rightarrow \begin{aligned} \psi_1 &= -\psi_3 \\ \psi_2 &= -\psi_4 \end{aligned} \quad \left. \vphantom{\begin{aligned} \psi_1 \\ \psi_2 \end{aligned}} \right\} \text{leaves 2 complex (4 real) degrees of freedom!}$$

WTF!?