Tuesday, February 23, 2016
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Let's actually solve the Dirac equation for a particle at rest, i.e. $\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial z} = 0 \Leftrightarrow \underbrace{\vec{p} = 0}_{i\hbar\partial_\mu \leftrightarrow p_\mu}$

Then we have: $\gamma^0\partial_0\psi + \frac{mc}{\hbar}\psi = 0$

$$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\frac{\partial\psi}{\partial ct} + \frac{mc}{\hbar}\psi = 0 \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Or:
$$\left.\begin{aligned} -\frac{i}{c}\frac{\partial\psi_3}{\partial t} + \frac{mc}{\hbar}\psi_1 &= 0 \\ -\frac{i}{c}\frac{\partial\psi_4}{\partial t} + \frac{mc}{\hbar}\psi_2 &= 0 \\ -\frac{i}{c}\frac{\partial\psi_1}{\partial t} + \frac{mc}{\hbar}\psi_3 &= 0 \\ -\frac{i}{c}\frac{\partial\psi_2}{\partial t} + \frac{mc}{\hbar}\psi_4 &= 0 \end{aligned}\right\}$$

4 solutions:
$$\psi^{(1)}_{rest} = e^{-\frac{imc^2}{\hbar}t}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \psi^{(2)}_{rest} = e^{-\frac{imc^2}{\hbar}t}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\psi^{(3)}_{rest} = e^{\frac{imc^2}{\hbar}t}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \qquad \psi^{(4)}_{rest} = e^{\frac{imc^2}{\hbar}t}\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

From QM we expect the time dependence of a state to evolve as $e^{-i\frac{E}{\hbar}t}$, and for particles at rest $E = mc^2$ so $e^{-i\frac{mc^2}{\hbar}t}$ indicates the usual behavior.

What about the solutions w/ $e^{\frac{iE}{\hbar}t}$ time-dependence? These correspond to antiparticle states!
These were given an interpretation early on by Feynman and Stueckelberg as positive energy states moving backwards in time.
This has been given a more modern understanding, but remains a useful idea and is actually implemented in Feynman diagrams.
This explains the 4 d.o.f. in the Dirac equation, $\pm\frac{1}{2}$ for each particle and antiparticle

$$\boxed{\begin{aligned} &\text{Note: Recall that } S_z = \frac{\hbar}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} \text{ for 4-comp. } \psi \\ &\qquad\qquad = \frac{\hbar}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \\ &\text{Then: } S_z\psi^{(1)} = \frac{\hbar}{2}\psi^{(1)}, \quad S_z\psi^{(2)} = -\frac{\hbar}{2}\psi^{(2)} \\ &\qquad S_z\psi^{(3)} = -\frac{\hbar}{2}\psi^{(3)}, \quad S_z\psi^{(4)} = \frac{\hbar}{2}\psi^{(4)} \end{aligned}}$$

So the Dirac equation secretly knows about and describes both particle and antiparticle behavior.
We will get a clearer picture of how these are related when we study discrete symmetries.

It is both more informative and more useful in calculations to consider plane-wave solutions since these correspond to states of definite momentum which is typically what we put in to scattering events and detect coming out.

Taking a plane-wave ansätz: $\psi(x) = \underset{\text{overall normalization constant}}{A}\, e^{-ik_\mu x^\mu} u(k^\mu)$ for which $\partial_\mu \psi = -ik_\mu \psi$

The Dirac equation then becomes: $(i\hbar\gamma^\mu k_\mu - mc)\psi = 0$ which is algebraic!

After some work one can show that $k^\mu = \pm\frac{1}{\hbar}p^\mu$ and we are left with 4 solutions:

$$\psi^{(1)} = Ae^{\frac{ip_\mu x^\mu}{\hbar}}\begin{pmatrix} E/mc^2 - p_z/mc \\ -p_x/mc - ip_y/mc \\ 1 \\ 0 \end{pmatrix} \qquad \psi^{(2)} = Ae^{\frac{ip_\mu x^\mu}{\hbar}}\begin{pmatrix} -p_x/mc + ip_y/mc \\ E/mc^2 + p_z/mc \\ 0 \\ 1 \end{pmatrix}$$

$$\psi^{(3)} = Ae^{-\frac{ip_\mu x^\mu}{\hbar}}\begin{pmatrix} 0 \\ 1 \\ -p_x/mc + ip_y/mc \\ -E/mc^2 + p_z/mc \end{pmatrix} \qquad \psi^{(4)} = Ae^{-\frac{ip_\mu x^\mu}{\hbar}}\begin{pmatrix} 1 \\ 0 \\ -E/mc^2 - p_z/mc \\ -p_x/mc - ip_y/mc \end{pmatrix}$$

Note: Typically we write:

$$\psi^{(1)} = Ae^{\frac{ip_\mu x^\mu}{\hbar}} u^{(1)}$$
$$\psi^{(2)} = Ae^{\frac{ip_\mu x^\mu}{\hbar}} u^{(2)}$$
$$\psi^{(3)} = Ae^{-\frac{ip_\mu x^\mu}{\hbar}} v^{(1)}$$
$$\psi^{(4)} = Ae^{-\frac{ip_\mu x^\mu}{\hbar}} v^{(2)}$$

where $u^{(1)}, u^{(2)}$ are particle spinors and $v^{(1)}, v^{(2)}$ are anti-particle spinors

Note: $\psi^{(i)} \to \psi^{(i)}_{rest}$ w/ $\vec{p} = 0$ (you fill in the details in the HW)

These are quite different than the "decoupled" spinors in NR QM, i.e. $\psi(x)\chi$ w/ $\chi = \alpha\binom{1}{0} + \beta\binom{0}{1}$.
Here the energy and momentum dependence cannot be extracted as an overall coefficient like $\psi(x)$.

Looking again at S_z, we note: $S_z\psi^{(1)} = \frac{\hbar}{2}Ae^{\frac{ip_\mu x^\mu}{\hbar}}\begin{pmatrix} E/mc^2 - p_z/mc \\ p_x/mc + ip_y/mc \\ 1 \\ 0 \end{pmatrix} \not\propto \psi^{(1)}$ So this and the other $\psi^{(i)}$ are not eigenstates of S_z.

Unless we choose $\vec{p} = p_z\hat{k}$, then: $S_z\psi^{(1)} = \frac{\hbar}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} Ae^{\frac{ip_\mu x^\mu}{\hbar}}\begin{pmatrix} E/mc^2 - p_z/mc \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\psi^{(1)}$

So it is often useful to work in terms of eigenstates of spin along the direction of motion ($\hat{k}$ above).
These are referred to as helicity states.

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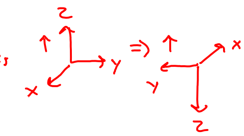
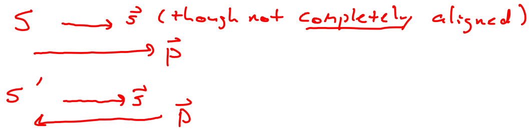
Characterizing particle states w/ helicity is <u>almost</u> just like characterizing them by S_z.

For instance if a particle has $S_z = +\frac{\hbar}{2}$, we can always rotate our coordinates so that the same particle has $S_z = -\frac{\hbar}{2}$. $\mathcal{I}$+ is still a useful classification if we stick to one coordinate system.



Helicity is similar. If we have $S_{\vec{p}} = +\frac{\hbar}{2}$ then
But we can always boost to frame reversing $\vec{p}$ then
giving us $S_{\vec{p}} = -\frac{\hbar}{2}$.



Except when the particle in question is massless! In that case there is no way to reverse $\vec{p}$ with a boost. So for massless particles, their helicity is an unchangeable intrinsic property (just like their total spin).

In fact for a given massless particle type (flavor) we might as well think of the $S_{\vec{p}} = \pm\frac{\hbar}{2}$ states as different particles!

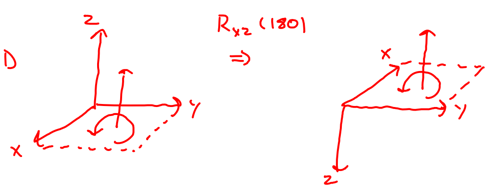
This has many implications, but first let's go back to our counting of states à la Wigner.

Recall we classify intrinsic spin states by the transformations that leave P^μ invariant.
For $m>0$, we can work w/ $P^\mu = (mc, 0, 0, 0) \Rightarrow$ 3D rotations $\Rightarrow$ spin-$\frac{1}{2}$ $\Rightarrow$ 2 states.
However for $m=0$ there is no rest frame. There is a simple P^μ to work with (remember the counting is independent of P^μ so we can choose any one that is handy).

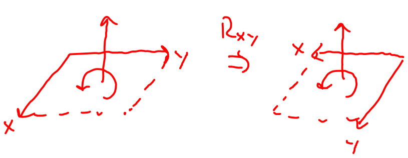
Consider: $P^\mu = (\frac{E}{c}, \frac{E}{c}, \underbrace{0, 0}_{})$ Note: $P_\mu P^\mu = 0$ as expected for $m=0$.

This is only invariant under 2D rotations! But these cannot change the spin in this plane!

Example: 3D



But in 2D



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Is any of this reflected in the Dirac equation?

Recall that w/ our conventions a boost on spinors is generated by $\sigma^{0i} = \frac{i}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$

while a rotation on spinors is generated by $\sigma^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$

So if we take our 4-component $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ we have that $\psi_\pm$ transform oppositely under boosts and alike under rotations.

The 2-component $\psi_\pm$ are called Weyl or chiral spinors.

The Dirac Lagrangian can be written: $\mathcal{L}_{Dirac} = (\hbar c)\bar{\psi}\gamma^\mu\partial_\mu\psi + mc^2\bar{\psi}\psi$

$$= -\hbar c\left(i\psi_-^\dagger \partial_\mu \sigma^\mu \psi_- + i\psi_+^\dagger \partial_\mu \bar{\sigma}^\mu \psi_+\right) + mc^2\left(\psi_-^\dagger\psi_+ + \psi_+^\dagger\psi_-\right)$$

$\sigma^\mu = (I, \sigma^i)$ $\qquad$ $\bar{\sigma}^\mu = (I, -\sigma^i)$

Recall: $\bar{\psi} = i\psi^\dagger\gamma^0$

$\gamma^0 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

So for $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \Rightarrow \bar{\psi} = (\psi_-^\dagger \ \psi_+^\dagger)$

Now the important thing to note is that if $m \neq 0$ we need both ψ_+ and ψ_- (hence a 4-comp. Dirac spinor). However if $m=0$, we can actually work with just one of ψ_+ or ψ_-, i.e. 2 component Weyl spinors which satisfy:

$$i\bar{\sigma}^\mu\partial_\mu\psi_+ = 0$$
or
$$i\sigma^\mu\partial_\mu\psi_- = 0$$

} Weyl equations. Each one describes a particle/anti-particle pair, hence 2 real d.o.f.

Choosing to work with ψ_+ or ψ_- for massless spinors is exactly the same as working with positive or negative helicity states! To see this consider one example:

$$\psi^{(1)} = A e^{\frac{i p_\mu x^\mu}{\hbar}}\begin{pmatrix} \frac{E}{mc^2} - \frac{p_z}{mc} \\ -\frac{p_x}{mc} - \frac{i p_y}{mc} \\ 1 \\ 0 \end{pmatrix} \xrightarrow[\vec{p} = p\hat{k}]{} \underbrace{A e^{\frac{i p_\mu x^\mu}{\hbar}}\begin{pmatrix} \frac{E}{mc^2} - \frac{p}{mc} \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{Eigenstate of } S_{\hat{p}}} \xrightarrow[m=0]{} \underbrace{A e^{\frac{i p_\mu x^\mu}{\hbar}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{Eigenstate of } S_{\hat{p}} \text{ and pure chirality } \begin{pmatrix} 0 \\ \psi_- \end{pmatrix}}$$

$m=0 \Downarrow E^2 - p^2c^2 = 0 \Downarrow E = pc$

since $S_{\hat{p}} = S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$

But not a state of pure chirality, e.g. $\begin{pmatrix} \psi_+ \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ \psi_- \end{pmatrix}$

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Everything we have done so far has been illustrated with our conventions for the γ's, but it may not be obvious that it works for other choices, e.g. when the γ's do not split like $-i\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$.

This is where γ^5 enters. For our conventions $\gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (Recall $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$)

And we can use it to form projection operators $P_\pm = \frac{1}{2}(1 \pm \gamma^5) \Rightarrow P_+\psi = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \psi_+$

$P_-\psi = \psi_-$

But from the definition of γ^5, we can show that $\frac{1}{2}(1 \pm \gamma^5)$ is a projection operator in any representation of γ's.

So instead of using the (representation dependent) split $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ we can just <u>define</u> $\psi_+ = P_+\psi$, $\psi_- = P_-\psi$

Here again we can see the difference between chirality and helicity:

$P_\pm = \frac{1}{2}(1 \pm \gamma^5)$ projects onto states of definite chirality.