Now that we have some insight into the Lagrangians and equations of motion for free scalar, spinor and vector fields, it is time to introduce interactions.

Recall that the free (or kinetic) terms typically involve derivatives (velocities) as well as mass terms (rest mass energy) which are <u>always</u> $\underbrace{\text{quadratic in the field in question.}}_{\text{This will be important to remember later on!}}$

$$\mathcal{L}_0 = \tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi + \tfrac{1}{2}\left(\tfrac{mc}{\hbar}\right)^2 \phi^2$$

$$\mathcal{L}_{1/2} = \hbar c \bar{\psi}\gamma^\mu \partial_\mu \psi + mc^2 \bar{\psi}\psi$$

$$\mathcal{L}_1 = \tfrac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{8\pi}\left(\tfrac{mc}{\hbar}\right)^2 A^\mu A_\mu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

What would interactions look like? Well generally, if 2 different fields interact, then there should be some term in the Lagrangian which multiplies the fields (or their derivatives), e.g. $\psi\phi$.

A priori it may seem that we could add any interaction terms we like (as long as they are Lorentz invariant) and in a certain deep sense (to be discussed much later) we should.

But it turns out that we can describe every experimental observation by only introducing interaction terms that follow from an elegant symmetry principle, that of local gauge invariance.

The general outline of the program is:

1. Begin with a free matter (Dirac) Lagrangian with some global symmetry.
2. Promote the global symmetry to a local gauge symmetry with the addition of a compensating gauge field (that itself must transform in a specific way). This will induce <u>interactions</u>.
3. Allow the gauge field to propagate by adding a field strength (kinetic) term to the Lagrangian.
4. Go have a beer!

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Example:

It might seem weird to have 2 constants, but in just a minute we will treat them differently!

$$\mathcal{L} = \hbar c \bar{\psi}\gamma^\mu \partial_\mu \psi + mc^2 \bar{\psi}\psi \qquad \bar{\psi} = i\psi^\dagger \gamma^0$$

some constant

parameter ϕ which is constant

Due to ψ being complex we could consider: $\psi \to \psi' = e^{iq\phi}\psi \Rightarrow \bar{\psi} \to \bar{\psi}' = \bar{\psi}e^{-iq\phi}$

$$\mathcal{L} \to \mathcal{L}' = \hbar c \bar{\psi} e^{-iq\phi}\gamma^\mu \partial_\mu e^{iq\phi}\psi + mc^2 \bar{\psi}e^{-iq\phi}e^{iq\phi}\psi = \mathcal{L}$$

We could do this because $e^{iq\phi}$ is constant.

So our starting point enjoys invariance under global (same everywhere) transformations by $e^{iq\phi}$.

But suppose we wanted to do this locally (different transformations at different locations)? $e^{iq\phi(x^\mu)}$

still constant

varies w/ position

We still have: $\psi \to \psi' = e^{iq\phi(x^\mu)}\psi \Rightarrow \bar{\psi} \to \bar{\psi}' = \bar{\psi}e^{-iq\phi(x^\mu)}$

$$\mathcal{L} \to \mathcal{L}' = \hbar c \bar{\psi} e^{-iq\phi(x^\mu)}\gamma^\mu \partial_\mu e^{iq\phi(x^\mu)}\psi + mc^2 \bar{\psi}\underbrace{e^{-iq\phi(x^\mu)}e^{iq\phi(x^\mu)}}_{I}\psi$$

We can't move this across ∂_μ!

So as written our Lagrangian is not invariant under local transformations. But we can make it!

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Let $\partial_\mu \Rightarrow D_\mu = \partial_\mu + i\underbrace{q}_{\text{same } q \text{ as before}} \overbrace{A_\mu}^{\text{A new field}}$

Then: $\mathcal{L} = \hbar c\bar\psi\gamma^\mu D_\mu\psi + mc^2\bar\psi\psi = \hbar c\bar\psi\gamma^\mu\partial_\mu\psi + iq\hbar c\bar\psi\gamma^\mu A_\mu\psi + mc^2\bar\psi\psi$

Now if we transform $\psi \to \psi' = e^{iq\phi(x^\mu)}\psi \Rightarrow \bar\psi \to \bar\psi' = \bar\psi e^{-iq\phi(x^\mu)}$ and $A_\mu \to A_\mu'$

$$\mathcal{L} \to \mathcal{L}' = \hbar c\bar\psi e^{-iq\phi(x^\mu)}\gamma^\mu \underbrace{\partial_\mu e^{iq\phi(x^\mu)}}_{\text{use product rule}}\psi + iq\hbar c\bar\psi \underbrace{e^{-iq\phi(x^\mu)}\gamma^\mu A_\mu' e^{iq\phi(x^\mu)}}_{I}\psi + mc^2\bar\psi \underbrace{e^{-iq\phi(x^\mu)}e^{iq\phi(x^\mu)}}_{I}\psi$$

$$\mathcal{L}' = \hbar c\bar\psi e^{-iq\phi(x^\mu)}\gamma^\mu\left[iq(\partial_\mu\phi(x^\mu))e^{iq\phi(x^\mu)}\psi + e^{iq\phi(x^\mu)}\partial_\mu\psi\right] + iq\hbar c\bar\psi\gamma^\mu A_\mu'\psi + mc^2\bar\psi\psi$$

$$= \hbar c\bar\psi\gamma^\mu\partial_\mu\psi + iq\hbar c\bar\psi\gamma^\mu(\partial_\mu\phi(x^\mu))\psi + iq\hbar c\bar\psi\gamma^\mu A_\mu'\psi + mc^2\bar\psi\psi$$

Now if $A_\mu' = A_\mu - \partial_\mu\phi(x^\mu)$ then:

$$\mathcal{L}' = \hbar c\bar\psi\gamma^\mu\partial_\mu\psi + iq\hbar c\underbrace{\bar\psi\gamma^\mu A_\mu\psi} + mc^2\bar\psi\psi = \mathcal{L}$$ BOOM!! It's invariant!

Look at what we got... an interaction $\bar\psi A_\mu\psi$.

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So far the new field A_μ cannot propagate since it has no kinetic term (it constitutes a background).

To give A_μ a kinetic term we could look to the Proca Lagrangian: $\mathcal{L}_1 = \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + \left(\frac{1}{8\pi}\right)\left(\frac{\overbrace{m}^{\text{mass of } A_\nu} c}{\hbar}\right)^2 A_\nu A^\nu$
w/ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

But we have to make sure that this new term is invariant under our local transformation.

Recall: $A_\mu \to A_\mu' = A_\mu - \partial_\mu \phi(x^\mu)$ (or $A_\nu \to A_\nu' = A_\nu - \partial_\nu \phi(x^\mu)$ etc.)

$$\mathcal{L}_1 \to \mathcal{L}' = \frac{1}{16\pi}(\partial_\mu A_\nu' - \partial_\nu A_\mu')\eta^{\mu\lambda}\eta^{\nu\rho}(\partial_\lambda A_\rho' - \partial_\rho A_\lambda') + \left(\frac{1}{8\pi}\right)\left(\frac{mc}{\hbar}\right)^2 A_\nu' \eta^{\mu\nu} A_\mu'$$

$$= \frac{1}{16\pi}(\partial_\mu A_\nu - \cancel{\partial_\mu \partial_\nu \phi} - \partial_\nu A_\mu + \cancel{\partial_\nu \partial_\mu \phi})\eta^{\mu\lambda}\eta^{\nu\rho}(\partial_\lambda A_\rho - \cancel{\partial_\lambda \partial_\rho \phi} - \partial_\rho A_\lambda + \cancel{\partial_\rho \partial_\lambda \phi})$$

$$+ \left(\frac{1}{8\pi}\right)\left(\frac{mc}{\hbar}\right)^2 \underbrace{(A_\nu - \partial_\nu \phi)\eta^{\mu\nu}(A_\mu - \partial_\mu \phi)}_{\text{This is not invariant!}}$$

So we can use Proca, but we must have $m_{A_\mu} = 0$!

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In total we have:

$$\mathcal{L} = \hbar c \bar{\psi} \gamma^\mu \partial_\mu \psi + iq\hbar c \bar{\psi} \gamma^\mu A_\mu \psi + mc^2 \bar{\psi}\psi + \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \quad \text{w/} \quad \overbrace{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}^{\text{The electromagnetic field strength.}}$$

This is invariant under local $\underbrace{e^{iq\phi(x^\mu)}}_{U^\dagger U = I}$ or $U(1)$ transformations. This is electromagnetism!

A_μ is the electromagnetic 4-vector potential which transforms under this gauge symmetry as $A_\mu \to A_\mu - \partial_\mu \phi$.
In this context we call A_μ a gauge field which mediates the electromagnetic interaction.

We can now identify q as the coupling strength of ψ to the electromagnetic field, i.e. charge!

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We will extend this to non-abelian symmetries in the days to come. But before that, a few clarifications will help.

When trying to make the symmetry local, the main problem we found was that the derivative did not satisfy the property we needed.

$$\bar{\psi}\gamma^\mu\partial_\mu\psi \to \bar{\psi}e^{-iq\phi}\gamma^\mu\partial_\mu e^{iq\phi}\psi \neq \bar{\psi}e^{-iq\phi}\gamma^\mu e^{iq\phi}\partial_\mu\psi$$

We fixed this with $D_\mu = \partial_\mu + iqA_\mu$ which in the end does do this:

$$\bar{\psi}\gamma^\mu D_\mu\psi \to \bar{\psi}e^{-iq\phi}\gamma^\mu D'_\mu e^{iq\phi}\psi = \bar{\psi}e^{-iq\phi}\gamma^\mu e^{iq\phi}D_\mu\psi = \bar{\psi}\gamma^\mu D_\mu\psi$$

So $D_\mu\psi \to D'_\mu e^{iq\phi}\psi = e^{iq\phi}D_\mu\psi$ is the magic property we needed.

This is equivalent to what is done in general relativity where the ordinary derivative ∂_μ proves to be nontensorial under a general coordinate transformation $x^\mu \to x^{\mu'}(x^\mu)$. To fix this we introduced the <u>covariant</u> derivative $\partial_\mu \to D_\mu + \underbrace{\Gamma\ldots}$

Christoffel connections play the role of gauge fields in GR.

So there is a close analogy to GR. But it gets better (and insightful for our purposes later).

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A different route to getting a gauge invariant field strength is to consider:

$$[D_\mu, D_\nu]\psi = (\partial_\mu + igA_\mu)(\partial_\nu + igA_\nu)\psi - (\partial_\nu + igA_\nu)(\partial_\mu + igA_\mu)\psi$$

$$= (\partial_\mu + igA_\mu)(\partial_\nu\psi + igA_\nu\psi) - (\partial_\nu + igA_\nu)(\partial_\mu\psi + igA_\mu\psi)$$

$$= \partial_\mu\partial_\nu\psi + igA_\mu\partial_\nu\psi - g^2A_\mu A_\nu\psi + ig(\partial_\mu A_\nu)\psi + igA_\nu\partial_\mu\psi$$

$$- \partial_\nu\partial_\mu\psi - igA_\nu\partial_\mu\psi + g^2A_\nu A_\mu\psi - ig(\partial_\nu A_\mu)\psi - igA_\mu\partial_\nu\psi$$

$$= ig(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi$$

So $F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu]$ which will be very helpful in the non-abelian case.

In GR we use $[D_\mu, D_\nu]V^\lambda = R^\lambda{}_{\mu\nu\alpha}V^\alpha$
to define the Riemann curvature tensor, i.e. the field strength of gravity!