Tuesday, March 22, 2016
4:59 AM

So far...

Free Lagrangians:

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^2 \qquad \text{K-G}$$

$$\mathcal{L}_{1/2} = \hbar c\,\bar{\psi}\gamma^\mu\partial_\mu\psi + mc^2\bar{\psi}\psi \qquad \text{Dirac}$$

$$\mathcal{L}_1 = \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 A_\nu A^\nu \qquad \text{Proca}$$

All imply $\frac{E^2}{c^2} - p^2 = m^2c^2 = -p_\mu p^\mu$

We start with spin-$\frac{1}{2}$ matter and introduce interactions via local gauge invariance:

$\partial_\mu \to D_\mu = \partial_\mu + ig\,\lambda\cdot A_\mu$ (use this new derivative in Dirac above)

To allow the new gauge fields to propagate we add:

$\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$ w/ $F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu]$

E&M: $U(1)$ on complex ψ

QCD: $SU(3)$ on $\psi = \begin{pmatrix}\psi_r\\ \psi_b\\ \psi_g\end{pmatrix}$

EW: $\underbrace{SU(2)_L}_{\text{acts on } \chi_L} \times \underbrace{U(1)_Y}_{\text{acts on } \chi_L \text{ and } e_R}$ on $\chi_L = \begin{pmatrix}\nu_e\\ e\end{pmatrix}_L$, etc. e_R, etc.

Monday, April 23, 2012
1:30 PM

The Higgs Mechanism

Last time we developed the electroweak interaction in terms of a true local gauge symmetry. As usual, when adding a kinetic (Proca) term for the gauge fields we were forced to take them to be massless since the Proca mass term $\left(\frac{Mc}{\hbar}\right)^2 A_\nu A^\nu$ is not gauge invariant. Moreover, to give matter a mass term we have to be able to combine left and right chiral spinors $mc^2(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$, but this clearly won't be invariant under $SU(2)_L$.

This one has an interesting resolution. To avoid unnecessary complications let's consider a simpler version based on $U(1)$.

(Instead of $SU(2)_L \times U(1)_Y$ which is the real story but much nastier!)

$$\mathcal{L} = \underbrace{\tfrac{1}{2}(\partial_\mu\phi)^*(\partial^\mu\phi)}_{\text{usual spin-0 kinetic}} \overbrace{\underbrace{-\tfrac{1}{2}\mu^2\phi^*\phi}_{\text{"wrong sign" mass term}} + \underbrace{\tfrac{1}{4}\lambda^2(\phi^*\phi)^2}_{\text{quartic self-interaction}}}^{\text{Let's just call this } U(\phi,\phi^*)} \qquad \text{where } \phi = \phi_1 + i\phi_2$$

This may seem interesting since $\mu^2 < 0 \Rightarrow \frac{E^2}{c^2} - p^2 < 0 \Rightarrow \underbrace{\gamma^2 m^2 v^2}_{p^2} > \underbrace{\gamma^2 m^2 c^2}_{\frac{E^2}{c^2} = \frac{(\gamma mc^2)^2}{c^2}} \Rightarrow v^2 > c^2 \Rightarrow$ tachyonic ?!
BUT... we will learn how to interpret this in a more sensible way in field theory.

Since ϕ is complex, we notice that this $\mathcal{L}$ has a global $U(1)$ symmetry, so we can play the familiar gauging game:

1. $\phi \to e^{i\theta}\phi \Rightarrow \phi^* \to e^{-i\theta}\phi^* \Rightarrow \mathcal{L} \to \mathcal{L}$
2. $\phi \to e^{i\theta(x)}\phi(x) \Rightarrow \partial_\mu \to D_\mu \equiv \partial_\mu + i\frac{q}{\hbar c}A_\mu$ w/ $A_\mu \to A_\mu' = A_\mu - \frac{\hbar c}{q}\partial_\mu\theta$
3. Add $\mathcal{L}_{Proca}$ w/ $m=0$ for A_μ to obtain:

$$\mathcal{L} = \tfrac{1}{2}\left[(\partial_\mu - \tfrac{iq}{\hbar c}A_\mu)\phi^*\right]\left[(\partial^\mu + \tfrac{iq}{\hbar c}A^\mu)\phi\right] - \tfrac{1}{2}\mu^2\phi^*\phi + \tfrac{1}{4}\lambda^2(\phi^*\phi)^2 + \tfrac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

So far we have a gauge theory w/ a massless gauge field.

However when we "do" particle physics what we are really studying are small fluctuations in the fields (one for each type of particle). But what is the larger "background" configuration of the fields (above which we study fluctuations)? We may assume it is zero, but is that consistent and are there other options?

We get the background field configurations by solving the classical e.o.m. from $\mathcal{L}$, then treat the small fluctuations quantum mechanically.

This is what underlies the spirit of Feynman diagrams, i.e. perturbative QFT. Backgrounds are decidedly non-perturbative.

Okay so let's see this in action:

If $\mathcal{L} = T + U(\phi, \phi^*)$ then the simplest solutions for backgrounds come from setting: $\partial\phi \to 0 \Rightarrow T = 0$
then: $\frac{\partial U}{\partial \phi^*} = 0$ solves e.o.m.

$$\mathcal{L}(\phi, \phi^*, A_\mu) = \underbrace{\tfrac{1}{2}\left[\left(\partial_\mu - \tfrac{iq}{\hbar c} A_\mu\right)\phi^*\right]\left[\left(\partial^\mu + \tfrac{iq}{\hbar c} A^\mu\right)\phi\right]}_{T_\phi} \underbrace{- \tfrac{1}{2}\mu^2 \phi^*\phi + \tfrac{1}{4}\lambda^2 (\phi^*\phi)^2}_{U(\phi, \phi^*)} + \underbrace{\tfrac{1}{16\pi} F^{\mu\nu}F_{\mu\nu}}_{T_{A^\nu}}$$

$$\frac{\partial U}{\partial \phi^*} = -\tfrac{1}{2}\mu^2\phi + \tfrac{1}{2}\lambda^2 \phi^*\phi\,\phi = -\tfrac{1}{2}\mu^2\phi + \tfrac{1}{2}\lambda^2 |\phi|^2 \phi$$

i) One solution is $A_\mu = 0$, $\phi = 0$

Using this solution and studying $\phi(x) = 0 + \delta\phi(x)$
$A_\mu(x) = 0 + \delta A_\mu(x)$

We have: $\mathcal{L}(\delta\phi, \delta\phi^*, \delta A_\mu) = \frac{1}{2}\left[\left(\partial_\mu - \frac{iq}{\hbar c}\delta A_\mu\right)\delta\phi^*\right]\left[\left(\partial^\mu + \frac{iq}{\hbar c}\delta A^\mu\right)\delta\phi\right] - \frac{1}{2}\mu^2 \delta\phi^*\delta\phi + \frac{1}{4}\lambda^2(\delta\phi^*\delta\phi) + \frac{1}{16\pi}\underbrace{F^{\mu\nu}F_{\mu\nu}}_{\text{from } \delta A_\mu}$

This looks exactly like what we had before, just w/ $\phi \to \delta\phi$, $\phi^* \to \delta\phi^*$, $A_\mu \to \delta A_\mu$.

ii) But another is $A_\mu = 0$, $\phi = \phi_0$ where ϕ_0 satisfies $|\phi_0|^2 = \frac{\mu^2}{\lambda^2} = \phi_{10}^2 + \phi_{20}^2$ (since $-\frac{1}{2}\mu^2\phi_0 + \frac{1}{2}\lambda^2|\phi_0|^2\phi_0 = 0$)

Using the specific choice $\phi_{10} = \frac{\mu}{\lambda}$, $\phi_{20} = 0$, $A_\mu = 0$ and studying $\phi_1(x) = \frac{\mu}{\lambda} + \delta\phi_1(x) \equiv \frac{\mu}{\lambda} + \eta(x)$, $\phi_2(x) = 0 + \delta\phi_2(x) \equiv \beta(x)$, $A_\mu(x) = 0 + \delta A_\mu(x) \equiv A_\mu(x)$ we find:

$$\mathcal{L} = \left[\tfrac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \mu^2\eta^2\right] + \left[\tfrac{1}{2}(\partial_\mu\beta)(\partial^\mu\beta)\right] + \left[+\tfrac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}\left(\tfrac{q}{\hbar c}\tfrac{\mu}{\lambda}\right)^2 A_\mu A^\mu\right]$$
$$+ \left\{\tfrac{q}{\hbar c}\left[\eta(\partial_\mu\beta) - \beta(\partial_\mu\eta)\right]A^\mu + \tfrac{\mu}{\lambda}\left(\tfrac{q}{\hbar c}\right)^2\eta(A_\mu A^\mu) + \tfrac{1}{2}\left(\tfrac{q}{\hbar c}\right)^2(\beta^2 + \eta^2)A_\mu A^\mu\right.$$
$$\left. + \lambda\mu(\eta^3 + \eta\beta^2) + \tfrac{1}{4}\lambda^2(\eta^4 + 2\eta^2\beta^2 + \beta^4)\right\} + \left(\tfrac{\mu}{\lambda}\tfrac{q}{\hbar c}\right)(\partial_\mu\beta)A^\mu - \left(\tfrac{\mu^2}{2\lambda}\right)^2$$

What does this describe?

- A massive real scalar field η w/ $m_\eta = \frac{\sqrt{2}\hbar}{c}\mu$
- A massive gauge field A_μ w/ $m_A = 2\sqrt{\pi}\left(\frac{q\mu}{\lambda c^2}\right)$ Boom!!
- A massless scalar β
- All are interacting w/ each other in weird ways.