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So far in our discussion of rotations in 3D we have encountered scalars, vectors and tensors, these are 0, 3 and higher dimensional representations of the rotation group.

What about a 2D representation of 3D rotations?

We would need 2×2 matrices satisfying $[g_i, g_j] = i\epsilon^{ijk} g_k$ where $\underbrace{i,j,k = 1,2,3}_{3D}$

$$[g_1, g_2] = i g_3$$
$$[g_2, g_3] = i g_1$$
$$[g_3, g_1] = i g_2$$

These work: $g_{R_{yz}} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} = \frac{1}{2}\sigma_x$ $\quad g_{R_{zx}} = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix} = \frac{1}{2}\sigma_y$ $\quad g_{R_{xy}} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} = \frac{1}{2}\sigma_z$ where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli spin matrices

Now we can build: $R_{yz}(\theta) = e^{i g_{R_{yz}} \theta} = \underbrace{\begin{pmatrix} \cos(\frac{\theta}{2}) & i\sin(\frac{\theta}{2}) \\ i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}}$ and similarly for R_{zx} and R_{xy}.

$$\left.\begin{matrix} \text{satisfy} & U^{+}U = \mathbb{I} \\ \text{and} & \det U = +1 \end{matrix}\right\} SU(2)$$ which act on complex 2-component <u>spinors</u> χ.

Often we write $\chi \to \chi' = e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\chi$ $\qquad$ Note: We will <u>not</u> use spin indices in this class, so we will rely on matrix manipulations.

So $SO(3) \sim SU(2)$, at least near the identity (which is all the Lie algebra knows about).

Globally however there is a difference:

$$\left.\begin{matrix} SO(3) & R_x(2\pi) = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix} = \mathbb{I} \\ SU(2) & R_x(2\pi) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbb{I} \end{matrix}\right\}$$ SU(2) is called the double-cover of SO(3)

of course $R_x(4\pi) = \mathbb{I}$ for both!

There is a certain sense in which spinors and SU(2) probes geometry more deeply than coordinates, scalars, vectors, SO(3), etc.

By "probe more deeply" I mean they contain more information. Sometimes people say that spinors know about the square root of the geometry.

In fact if we consider the anti-commutator of the Pauli matrices we find: $\underbrace{\{\sigma_i, \sigma_j\} = 2\delta_{ij} I_{2\times2}}^{\text{A Clifford algebra.}}$

Example: $\sigma_x\sigma_y + \sigma_y\sigma_x = \begin{pmatrix} 0&1 \\ 1&0 \end{pmatrix}\begin{pmatrix} 0&-i \\ i&0 \end{pmatrix} + \begin{pmatrix} 0&-i \\ i&0 \end{pmatrix}\begin{pmatrix} 0&1 \\ 1&0 \end{pmatrix} = \begin{pmatrix} i&0 \\ 0&-i \end{pmatrix} + \begin{pmatrix} -i&0 \\ 0&i \end{pmatrix} = \begin{pmatrix} 0&0 \\ 0&0 \end{pmatrix}$ as expected since $\delta_{xy} = 0$

$\sigma_y\sigma_y + \sigma_y\sigma_y = 2\sigma_y\sigma_y = 2\begin{pmatrix} 0&-i \\ i&0 \end{pmatrix}\begin{pmatrix} 0&-i \\ i&0 \end{pmatrix} = 2\begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$ as expected since $\delta_{yy} = 1$

It might seem silly, but recall that $\delta_{ij} = \begin{pmatrix} 1 && \\ &1& \\ &&1 \end{pmatrix}$ is the metric of $\mathbb{R}^3$. This will come in handy later.

Another illustration of this is a lesson from QM: If we only have integer spin states at our disposal, {0, 1, 2, ...} then by combining spins we can only ever build more integer spin states. However if we allow ½ integer spin states, then we can build ½ or whole integer states <u>just using</u> ½ spin states, e.g. $\frac{1}{2} - \frac{1}{2} = 0$, $\frac{1}{2} + \frac{1}{2} = 1$.

To finish up, we need to determine how to build an invariant (for Lagrangians) out of spinors.

Following our usual recipe: If $\chi \rightarrow \chi' = e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\chi$ and $\tilde{\chi} \rightarrow \tilde{\chi}' = (e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}})^{-1\dagger}\tilde{\chi}$

then $\tilde{\chi}^{\dagger \xleftarrow{} *T}\chi$ is invariant.

But recall how we form $\tilde{\chi}$ from χ: $\tilde{\chi} = (g\chi)$ where $(e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}})^{\dagger} g\, e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}} = g$

However for SU(2) we already know that $U^{\dagger}U = 1$ so $g = I$ and

we can say $\tilde{\chi} = (g\chi) = \chi$ and then $\chi^{\dagger}\chi$ is invariant!

Note: All of the σ matrices are Hermitian, i.e. $\sigma_i^{\dagger} = \sigma_i$, $\vec{\theta}$ is real so

$$U^{\dagger} = \underbrace{(e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}})^{\dagger} = e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}}_{\text{You can see more explicitly by Taylor expanding}} = U^{-1}$$

This will <u>not</u> be the case later!

$$\left[I + \left(\tfrac{i}{2}\vec{\sigma}\cdot\vec{\theta}\right) + \tfrac{1}{2}\left(\tfrac{i}{2}\vec{\sigma}\cdot\vec{\theta}\right)\left(\tfrac{i}{2}\vec{\sigma}\cdot\vec{\theta}\right) + \cdots\right]^{\dagger}$$

$$= I + \left(-\tfrac{i}{2}\vec{\sigma}^{\dagger}\cdot\vec{\theta}\right) + \tfrac{1}{2}\left(-\tfrac{i}{2}\vec{\sigma}^{\dagger}\cdot\vec{\theta}\right)\left(-\tfrac{i}{2}\vec{\sigma}^{\dagger}\cdot\vec{\theta}\right) + \cdots$$

$$= I + \left(-\tfrac{i}{2}\vec{\sigma}\cdot\vec{\theta}\right) + \tfrac{1}{2}\left(-\tfrac{i}{2}\vec{\sigma}\cdot\vec{\theta}\right)\left(-\tfrac{i}{2}\vec{\sigma}\cdot\vec{\theta}\right) = e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}$$

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Now it is time to repeat this procedure for special relativity.

The Lorentz transformations as they act on coordinates/vectors form $SO(1,3)$ so let's explore its algebra.

We expect 6 generators corresponding to: $\underbrace{R_{yz}, R_{zx}, R_{xy}}, \underbrace{B_{xt}, B_{yt}, B_{zt}}$.

We will call the corresponding generators: $J_1, J_2, J_3 \quad K_1, K_2, K_3$

Fortunately we already know a lot about the J's :

From which we can also get $SU(2)$

$$J_1 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-i \\ 0&0&i&0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&i \\ 0&0&0&0 \\ 0&-i&0&0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0&0&0&0 \\ 0&0&-i&0 \\ 0&i&0&0 \\ 0&0&0&0 \end{pmatrix} \Rightarrow \underbrace{[J_i, J_j] = i\epsilon^{ijk} J_k}$$

If we take the various boosts and again consider their Taylor expansion, then using the exponential map $B = \exp(i K \delta B)$ we find:

$$K_1 = \begin{pmatrix} 0&-i&0&0 \\ -i&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0&0&-i&0 \\ 0&0&0&0 \\ -i&0&0&0 \\ 0&0&0&0 \end{pmatrix} \quad K_3 = \begin{pmatrix} 0&0&0&-i \\ 0&0&0&0 \\ 0&0&0&0 \\ -i&0&0&0 \end{pmatrix}$$

Now is where it gets interesting. By brute force one can show:

$$[K_i, K_j] = -i\epsilon^{ijk} J_k \qquad \text{2 boosts} \rightarrow \text{rotation}$$

$$[J_i, K_j] = i\epsilon^{ijk} K_k \qquad \text{rotation + boost = boost}$$

Question: Can the boosts alone form a subgroup of $SO(1,3)$? No
What about rotations? Yup

So unfortunately the boosts and rotations of $SO(1,3)$ do not cleanly split from each other.

But...

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Let's play an old math/physics trick:

Define $J_{+i} = \frac{1}{2}(J_i + iK_i)$

$J_{-i} = \frac{1}{2}(J_i - iK_i)$

$$\left.\begin{array}{l} \\ \end{array}\right\} \Rightarrow \begin{array}{ll} [J_{+i}, J_{+j}] = i\epsilon^{ijk} J_{+k} & \Rightarrow SO(3) \\ {[J_{-i}, J_{-j}] = i\epsilon^{ijk} J_{-k}} & \Rightarrow SO(3) \\ {[J_{+i}, J_{-j}] = 0} & \Rightarrow \text{These } SO(3) \text{ don't mix.} \end{array}$$

Then:

So we find that at least near the identity $SO(1,3) \sim \underbrace{SO(3)\times SO(3)}$

Remember this is **not** a split into 3 boosts and 3 rotations!!

Now everything so far has been in terms of coordinates (scalars, vectors, tensors, etc.), but we can immediately see how to introduce spinors.

We utilize $SO(1,3) \sim SO(3)\times SO(3) \sim \underbrace{SU(2)\times SU(2)}$

Each of these will act on a complex 2 component object, so our total spinor in 4D has 4 complex components!

This is most unfortunate since now we have 4 component vectors and 4 component spinors, but the components mean totally different things. This is only a misfortune in 4D.

	3D	4D	5D	6D	7D	8D	9D	10D
vector	3	4	5	6	7	8	9	10
spinor	2	4	4	8	8	16	16	32

The counting goes: For each *independent* plane you can define an independent $SU(2)$ w/ a 2-component spinor giving $2^{d/2}$ or $2^{(d-1)/2}$ states depending on d even or odd.