Friday, February 05, 2016
9:25 AM

We now need to determine how these 4-component spinors transform and then how to build an invariant.

You might the we could just use the 4x4 matrices we already have for K_i and J_i, but remember these act on coordinate related quantities not spinors.

So what should we use? There are numerous ways to get at the answer, but we will use the deepest based on the idea of the square root of the geometry.

Recall: $\{\sigma_i, \sigma_j\} = 2\underbrace{\delta_{ij}}_{\text{metric in 3D}} I_{2\times2} \quad \Rightarrow \chi \to \chi' = e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\chi$

Then perhaps: $\{\underbrace{\gamma^\mu, \gamma^\nu}_{\gamma^0, \gamma^1, \gamma^2, \gamma^3}\} = 2\eta^{\mu\nu} I_{4\times4} \quad \overset{?}{\Rightarrow} \psi \to \psi' = e^{\frac{i}{2}\vec{\gamma}\cdot\vec{\theta}}\psi$

Unfortunately this won't work! We expect 4 γ's, so this is only 4 distinct transformations, but we know there should be 6!
↑
$SO(1,3)$

Fortunately the answer is hiding in our bad notation.

If instead of $(\sigma_1, \sigma_2, \sigma_3)$ we think of $(-i[\sigma_2, \sigma_3], -i[\sigma_3, \sigma_1], -i[\sigma_1, \sigma_2])$

Rotation around x is really in the y-z plane.

Then we can think of: $\left(-\frac{i}{4}[\gamma^0, \gamma^1], -\frac{i}{4}[\gamma^0, \gamma^2], -\frac{i}{4}[\gamma^0, \gamma^3], -\frac{i}{4}[\gamma^1, \gamma^2], -\frac{i}{4}[\gamma^2, \gamma^3], -\frac{i}{4}[\gamma^3, \gamma^1]\right)$

If we call these $\sigma^{\mu\nu} = \{\sigma^{01}, \sigma^{02}, \sigma^{03}, \sigma^{12}, \sigma^{23}, \sigma^{31}\}$
$\sigma^{01} = -\sigma^{10}$, $\sigma^{02} = -\sigma^{20}$, $\sigma^{03} = -\sigma^{30}$, $\sigma^{12} = -\sigma^{21}$, $\sigma^{23} = -\sigma^{32}$, $\sigma^{31} = -\sigma^{13}$

Then parameterizing the transformation with angles $\{\alpha, \beta, \gamma, \theta, \phi, \psi\} \equiv \{\omega_{01}, \omega_{02}, \omega_{03}, \omega_{12}, \omega_{23}, \omega_{31}\}$
$\omega_{01} = -\omega_{10}$, $\omega_{02} = -\omega_{20}$, $\omega_{03} = -\omega_{30}$, $\omega_{12} = -\omega_{21}$, $\omega_{23} = -\omega_{32}$, $\omega_{31} = -\omega_{13}$

We can write our transformation: $\boxed{\psi \to \psi' = e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi}$

Example: Rotation in y-z by ϕ uses $\omega_{\mu\nu} = \{0, 0, 0, 0, \omega_{23} = \phi, 0\}$ giving $\psi \to \psi' = e^{\frac{i}{4}(\sigma^{23}\omega_{23} + \sigma^{32}\omega_{32})}\psi$
or
$\psi' = e^{\frac{i}{2}\sigma^{23}\phi}\psi$

Monday, February 08, 2016
7:45 AM

Without any further ado, I present (at least one set of) the Dirac γ matrices:

$$\gamma^0 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = -i\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad \text{w/} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Example: $\gamma^2 = -i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$

These have some nice properties:

- Recall $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_{4\times 4}$
- Then $(\gamma^0)^2 = -1$, $(\gamma^i)^2 = 1$
- And $\underbrace{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0}_{\text{or } \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu}$ if $\mu \neq \nu$ since $\eta^{\mu\nu}$ is diagonal

We can now explicitly form the generators:

$$\sigma^{0i} = -\frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{4}\left[\gamma^0\gamma^i - \gamma^i\gamma^0\right]$$

$$= -\frac{i}{4}\left[-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(-i)\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} + i\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}(-i)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

$$= \frac{i}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$$

Note: We now see why we needed the $\frac{i}{4}$ in the definition. The transformation now reduces to the usual $SU(2)$ transformation on each pair of spinor indices, i.e. $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \begin{matrix} \} SU(2) \\ \} SU(2) \end{matrix}$

$$\sigma^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \qquad \text{e.g.} \quad \sigma^{12} = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

Monday, February 08, 2016
8:15 AM

Last time we tried to build an invariant $\tilde{\psi}^{\dagger}\psi$ from spinors trying $\tilde{\psi}=\psi$ like we did with $SU(2)$,

but...
$$\tilde{\psi}^{\dagger}\psi = \psi^{\dagger}\psi \to (\psi')^{\dagger}\psi' = \left(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi\right)^{\dagger} e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi$$
$$= \psi^{\dagger}\underbrace{\left(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\right)^{\dagger}} e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi$$

$$\boxed{\text{Recall: } \sigma^{0i} = \frac{i}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{pmatrix} \quad \sigma^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix} \quad \text{and } \sigma^{\mu\dagger} = \sigma^{\mu}}$$

But the $\sigma^{\mu\nu}$ are <u>not</u> all Hermitian!

In particular $\sigma^{0i\,\dagger} = -\sigma^{0i}$ Not-Hermitian
$\sigma^{ij\,\dagger} = \sigma^{ij}$ Hermitian

The reason $\psi^{\dagger}\psi$ worked for $SU(2)$ is that the generators were all Hermitian.

But we can fix this with a different choice of dual: $\tilde{\psi} = i\gamma^0\psi$

$\gamma^0 = -i\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$

Then: $\tilde{\psi}^{\dagger}\psi = (i\gamma^0\psi)^{\dagger}\psi$

$= -i\,\psi^{\dagger}\gamma^{0\dagger}\psi$

$= i\,\psi^{\dagger}\gamma^0\psi \to i(\psi')^{\dagger}\gamma^0\psi' = i\left(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi\right)^{\dagger}\gamma^0 e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi$

$$= i\psi^{\dagger}\underbrace{\left(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\right)^{\dagger}\gamma^0}_{\gamma^0 e^{-\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}} e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi$$

<u>You will show this in your HW</u>

Same! $\quad = i\psi^{\dagger}\gamma^0\psi$

So in the end we define our dual spinor with $\tilde{\psi} = i\gamma^0\psi$ and the adjoint $\bar{\psi} \equiv \tilde{\psi}^{\dagger} = i\psi^{\dagger}\gamma^0$

$\Downarrow$

$\bar{\psi}\psi$ is invariant

Or in other words $g = i\gamma^0 \Rightarrow \tilde{\psi} = g\psi$