Tuesday, February 16, 2016
8:36 AM

Now that we have an invariant we can include spinor degrees of freedom in an invariant Lagrangian.

But we can do more! The γ^μ matrices have the unique property of "linking" spin space to spacetime. Recall they secretly have 2 spin indices which we omit γ_{ab} (that's what makes them matrices) and they obviously have one spacetime index. We might wonder if this truly makes the γ^μ a vector in spacetime. More on that and what we can do with it in a moment.

Technical aside: Spin space is tied to the geometry at hand. In fact just as vectors actually live in the tangent space at each point, or tangent bundle over all of spacetime, spinors live in the spin bundle over spacetime.
Since both spinors and vectors are tied at some level to spacetime, if we transform coordinates both get impacted wherever such indices appear.
To that end we introduce some abbreviated notation: If: $V^\mu \to V^{\mu'} = \Lambda^{\mu'}{}_\mu V^\mu$
$$\psi \to \psi' = \underbrace{e^{\mp \sigma^{\mu\nu}\omega_{\mu\nu}}}\psi \longrightarrow S[\Lambda]$$
The spinor version of Λ

Back to γ^μ as a vector. This is one place where we will restore spinor indices. If we say ψ^a transforms as $\psi^a \to \psi^{a'} = S[\Lambda]^{a'}{}_a \psi^a$ and $\bar\psi_b \to \bar\psi_{b'} = S[\Lambda]^b{}_{b'} \bar\psi_b$.
But for $\bar\psi_a \psi^a$ to be invariant we need $\bar\psi_a \psi^a \to \bar\psi_{a'}\psi^{a'} = \bar\psi_a \underbrace{S[\Lambda]^a{}_{a'} S[\Lambda]^{a'}{}_b}_{=I^a{}_b} \psi^b$

Now the correct way to label γ^μ is $\gamma^{\mu a}{}_b$. This is so that $\bar\psi_a \gamma^{\mu a}{}_b \psi^b$ makes sense w/ the Einstein summation convention.

If we call $S[\Lambda]^{a'}{}_a = S[\Lambda]$ then $S[\Lambda]^a{}_{a'} = S[\Lambda]^{-1}$

To transform $\gamma^{\mu a}{}_b$ we simply transform every index in sight:
$$\gamma^{\mu a}{}_b \to \gamma^{\mu' a'}{}_{b'} = \Lambda^{\mu'}{}_\mu S[\Lambda]^{a'}{}_a S[\Lambda]^b{}_{b'} \gamma^{\mu a}{}_b = \Lambda^{\mu'}{}_\mu S[\Lambda]^{a'}{}_a \gamma^{\mu a}{}_b S[\Lambda]^b{}_{b'}$$

Now hiding the spinor indices this becomes: $\gamma^\mu \to \gamma^{\mu'} = \Lambda^{\mu'}{}_\mu S[\Lambda]\gamma^\mu S[\Lambda]^{-1}$

In practice we usually only use the vector index to transform. The reason is that whenever a γ^μ appears in say a Lagrangian, it is in the middle of a "spinor sandwich", e.g. $\bar\psi\gamma^\mu\psi$.

The need for spinor sandwiches should be somewhat apparent ... they have no overall spin indices, so are invariant spin-wise (but may still have vector indices to take care of!).

In fact: $\bar\psi\psi$ - scalar
$\bar\psi\gamma^\mu\psi$ - vector
$\bar\psi\gamma^\mu\gamma^\nu\psi$ - (2,0) tensor
etc.

$$\bar\psi\gamma^\mu\psi \to \bar\psi'\gamma^{\mu'}\psi' = \bar\psi S[\Lambda]^{-1} S[\Lambda]\Lambda^{\mu'}{}_\mu \gamma^\mu S[\Lambda]^{-1}S[\Lambda]\psi$$
$$= \Lambda^{\mu'}{}_\mu \underbrace{\bar\psi\gamma^\mu\psi}_{\text{transforms like a vector}}$$

You can think of spinor sandwiches in terms of matrix multiplication of spin indices:

$\bar\psi\psi \Rightarrow (\cdot\cdot\cdot\cdot)\begin{pmatrix}\cdot\\\cdot\\\cdot\end{pmatrix} = (\cdot)$

$\bar\psi\gamma^\mu\psi \Rightarrow (\cdot\cdot\cdot\cdot)\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}\cdot\\\cdot\\\cdot\end{pmatrix} = (\cdot)$

or insert as many matrices as you like!

Thursday, February 8, 2018
2:40 PM

Actions

Of the various formalisms for describing mechanics, the Lagrangian and associated action principle is the most useful for our purposes since it can be generalized in several ways:

$$\underbrace{NR \to \text{Relativistic}}_{\int dt \,\to\, \int d^4x}, \quad \underbrace{PP \to \text{Fields}}_{x(t) \to \phi(x^\mu)}, \quad \underbrace{CM \to QM}_{\delta S = 0 \,\to\, \int D\phi\, e^{\frac{i}{\hbar}S}}$$

and because it allows us to work with manifest symmetries.

Preliminary:

The action is a functional which essentially means a function of functions.

For example, whereas a function takes an argument and returns a number: $f(x)=x^2 \Rightarrow f(2)=4$
a functional does the same when we insert a function: $S[f(x)] = \int_0^1 f(x)dx$

$$\Downarrow$$

$$S[x^2] = \int_0^1 x^2 dx = \frac{1}{3}$$

What makes this nice is that in the same way we can extremize a function w/ $\frac{df}{dx}=0$ to find the value of x which achieves the max or min of $f(x)$, we can also extremize a functional w/ $\frac{\delta S}{\delta f}=0$ to find the function $f(x)$ which achieves the max or min of $S[f(x)]$.

You should be familiar w/ $S=\int L\,dt$ $\quad L(q,\dot{q}) = T-V$ and $\delta S = 0 \Rightarrow \underbrace{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0}_{\text{Euler-Lagrange e.o.m.}}$

We need a relativistic version of this suitable for quantum use.

Thursday, February 8, 2018
2:53 PM

Action, Lagrangian & E.O.M. for Relativistic Fields

Replace: $q(t) \to \phi(x^\mu)$, $\dot{q}(t) \to \frac{\partial \phi}{\partial x^\mu}$, $\int (\ ) dt \to \int (\ ) d^4x$

$$S = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x$$

Eventually we will incorporate this in a path integral, but it also turns out that the classical e.o.m. will be useful.

Consider $\phi(x^\mu)$ and a region M of spacetime w/ boundary conditions specified $\phi(x^\mu|_{\partial M})$

Now consider a deformed field configuration $\phi'(x^\mu) = \phi(x^\mu) + \delta\phi(x^\mu)$ that satisfies the same b.c.s., i.e. $\phi'(x^\mu|_{\partial M}) = \phi(x^\mu|_{\partial M}) \Rightarrow \delta\phi(x^\mu|_{\partial M}) = 0$

The classical field configuration satisfies $\delta S = \int \delta\mathcal{L}(\phi, \partial_\mu\phi)\, d^4x = 0$

$$\delta S = \int \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)} \right] = 0$$

I.B.P. using $\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) = \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\partial_\mu\phi$

$$\delta S = \int \left[\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]\delta\phi\, d^4x + \underbrace{\int \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) d^4x}_{= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi \big|_{\partial M} = 0 \text{ by}} = 0$$

For this to be true for arbitrary $\delta\phi$ we need: $\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0$ EL e.o.m.

compare to: $\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}}\right) = 0$

If you are familiar with Lagrangian mechanics then you are probably used to constructing the Lagrangian with the kinetic and potential energy of the degrees of freedom according to

$$L = \underbrace{T}_{\text{kinetic energy}} - \underbrace{V}_{\text{Potential energy}}$$

The idea of potential energy is useful, but for our purposes it is better to directly associate potential energies with the interactions between things (fields in our case).

So generically we expect the Lagrangian to split into: $\mathcal{L} = \mathcal{L}_{kinetic} + \mathcal{L}_{interaction}$

This sign is not that important since we haven't yet specified how to construct $\mathcal{L}_{interaction}$

Now if we have no interactions, then we have what is called a free theory in which case $\mathcal{L}_{free} = \mathcal{L}_{kinetic}$ so sometimes we call the kinetic term the "free Lagrangian".

The program we will follow is to first consider free Lagrangians and then introduce interactions through the principle of local gauge invariance.

Free Lagrangians

In classical point particle physics you would say $L_{free} = T = \frac{p^2}{2m} = \frac{1}{2}mv^2$.

However this is based on a massive scalar particle and built from the 3-momentum $\vec{p}$ (or velocity $\vec{v}$).

All of this changes for (possibly massless) relativistic fields.

Fortunately we only have 3 cases to consider: $\underbrace{\text{spin-}0}_{\text{Higgs}}$, $\underbrace{\text{spin-}\tfrac{1}{2}}_{\text{matter}}$, $\underbrace{\text{spin-}1}_{\text{force particles}}$

We won't derive these free Lagrangians. Doing so can be done in various ways at different levels of sophistication and to be honest several of them were actually guessed in their original discovery.
We will just present them one at a time, then derive the Euler-Lagrange equation of motion.

Wednesday, February 17, 2016
7:34 PM

Spin-0 (scalars) ϕ

$$\mathcal{L}_{Free} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^2$$

Notice that the mass term is separate which allows us to consistently handle $m=0$ cases.

$$= \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^2$$

$$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0$$

$$\left(\frac{mc}{\hbar}\right)^2\phi - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial}{\partial x^\nu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\right) = 0$$

$$\left(\frac{mc}{\hbar}\right)^2\phi - \frac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - \frac{1}{2}\eta^{\mu\nu}\partial_\nu\partial_\mu\phi = 0$$

$$\left(\frac{mc}{\hbar}\right)^2\phi - \eta^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$$

$$\partial_\mu\partial^\mu\phi - \left(\frac{mc}{\hbar}\right)^2\phi = 0$$ The Klein-Gordon Equation

Wednesday, February 17, 2016
7:41 PM

Spin-1 (vectors) A^μ

$$\mathcal{L}_{free} = \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 A^\mu A_\mu \quad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \text{ and } F^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\rho}F_{\lambda\rho}$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = \frac{1}{4\pi}\left(\frac{mc}{\hbar}\right)^2 A^\mu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = \frac{1}{4\pi} F^{\mu\nu}$$ (You get to fill in the details in your homework)

Then: $\partial_\mu F^{\mu\nu} - \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0$ The Proca Equation

Taking $m^2 = 0$ we have $\partial_\mu F^{\mu\nu} = 0 \Rightarrow$

$$\left.\begin{aligned} &\vec{\nabla}\times\vec{B} - \frac{\partial \vec{E}}{\partial t} = \overset{\text{usually } \vec{J}}{0} \\ &\vec{\nabla}\cdot\vec{E} = \underset{\text{usually } \rho}{0} \end{aligned}\right\} \text{ ½ of Maxwell's equations}$$

Note: $F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$

$$\left.\begin{aligned} &\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \\ &\vec{\nabla}\cdot\vec{B} = 0 \end{aligned}\right\} \text{ Non-dynamical, come from electromagnetic ``geometry''}$$

Wednesday, February 17, 2016
7:56 PM

<u>Spin-$\frac{1}{2}$ (spinors)</u> ψ & $\bar{\psi}$

$\mathcal{L}_{free} = (\hbar c)\bar{\psi}\gamma^\mu\partial_\mu\psi + mc^2\bar{\psi}\psi$ We treat ψ & $\bar{\psi}$ as independent degrees of freedom for reasons to be discussed.

Varying w.r.t. $\bar{\psi}$:

$$\underbrace{\frac{\partial\mathcal{L}}{\partial\bar{\psi}} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) = 0}$$

$$\hbar\underbrace{\gamma^\mu\partial_\mu}_{\equiv\not{\partial}}\psi + mc^2\psi = 0 \Rightarrow \not{\partial}\psi + \frac{mc}{\hbar}\psi = 0$$ <u>The Dirac Equation</u>

Interesting...

Varying w.r.t. ψ:

$$\frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = 0$$

$$mc^2\bar{\psi} - \partial_\mu(\hbar c\bar{\psi}\gamma^\mu) = 0 = \not{\partial}\bar{\psi} - \frac{mc}{\hbar}\bar{\psi} = 0$$ (The adjoint of the Dirac Equation)