**Short:** *(5 pts each)*

1. List two ways in which the wave-function of a quantum system can evolve.

a) Dynamically through the TDSE.

b) Collapse due to measurement.

2. Is the expectation value of a measurement the most probable outcome?

No. The expectation value would be the average result after "many" measurements on identical systems.

3. Are all solutions to the $T.I.S.E.$ probability amplitudes?

No. Only the properly normalized ones would be.

4. The World Wrestling Federation created the dynamical quantity "Smackdown" defined as $SD = (p^3 - kx)$ which measures the heartbreak felt by the unfortunate recipient of a bodyslam from the top rope, where $k$ is the smackdown constant. Write down an explicit expression for the expectation value of the quantum Smackdown operator $\hat{SD}$ for the quantum state $\psi$. Your answer should include only $i, k, \hbar, x, \psi, \partial, *$ and an integral sign.

$$\int_{-\infty}^{\infty} \psi^* \hat{SD} \psi \, dx = \int_{-\infty}^{\infty} \psi^* (\hat{p}^3 - kx) \psi \, dx = \int_{-\infty}^{\infty} \psi^* \left( \left(-i\hbar \frac{\partial}{\partial x}\right)^3 - kx \right) \psi \, dx$$

5. In a stationary state one always finds that $\langle \hat{p} \rangle = m \frac{d\langle \hat{x} \rangle}{dt} = 0$. Is this true for a general solution of the $T.D.S.E.$? Explain your answer.

No. If the general solution has more than one term in the sum, e.g. $\Psi = c_1\psi_1 + c_2\psi_2$ then the time dependence of expectation values will not cancel.

6. For a quantum system $\langle \hat{p}^2 \rangle = 2\hbar^2$ and $\langle \hat{x} \rangle = 3$. Write down an inequality that $\langle \hat{x}^2 \rangle$ must satisfy.

$\langle \hat{p} \rangle = m \frac{d\langle \hat{x} \rangle}{dt} = 0$

$$\sigma_x \sigma_p \geq \frac{\hbar}{2} \Rightarrow \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2} \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2} = \sqrt{\langle \hat{x}^2 \rangle - 9} \sqrt{2\hbar^2} \geq \frac{\hbar}{2} \Rightarrow \langle \hat{x}^2 \rangle \geq \frac{1}{8} + 9 = \frac{73}{8}$$

7. For a stationary state of a quantum system one calculates $\langle \hat{H} \rangle = 32\,J$. What is $\langle \hat{H}^2 \rangle$ for this system? If it cannot be determined, explain why.

For stationary states $\sigma_H = \sqrt{\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2} = 0 \Rightarrow \langle \hat{H}^2 \rangle = \langle \hat{H} \rangle^2 = 32^2\,J^2$

8. For an even potential $V(x)$) we can, without loss of generality, use an orthormal set of basis functions $\psi_n(x)$ which are even (for $n$-even) and odd (for $n$-odd) under $x \to -x$. With this basis is $\int_{-\infty}^{\infty} \psi_n^* \psi_{n+2} \, dx$ necessarily equal to 0? Yes.

For an orthormal basis $\int_{-\infty}^{\infty} \psi_n^* \psi_m \, dx = \delta_{nm}$ if $m = n+2$ then $\delta_{n\,n+2} = 0$.

**Tall:** *(15 pts each)* Choose one of each pair of questions below to answer.

9a. For a normalized wavefunction given by $\psi(x) = \begin{cases} \frac{1}{\sqrt{a}} sin\left(\frac{\pi}{a}x\right) & |x| \le a \\ 0 & |x| > a \end{cases}$ calculate $\langle \hat{p} \rangle$.

9b. For a normalized wavefunction $\psi(x) = \begin{cases} \frac{1}{\sqrt{a}} sin\left(\frac{\pi}{a}x\right) & |x| \le a \\ 0 & |x| > a \end{cases}$ and $V(x) = V_0$, find $E$.

9a. $\langle \hat{p} \rangle = \int_{-\infty}^{\infty} \psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\psi\,dx = \frac{1}{a}\int_{-a}^{a} \sin\left(\frac{\pi}{a}x\right)\left(i\hbar\frac{\pi^2}{a^2}\right)\cos\left(\frac{\pi}{a}x\right)dx$

$= i\hbar\frac{\pi^2}{a^3}\int_{-a}^{a} \underbrace{\sin\left(\frac{\pi}{a}x\right)}_{\text{Odd}}\underbrace{\cos\left(\frac{\pi}{a}x\right)}_{\text{Even}}dx = 0$

Symmetric

9b. $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = \frac{\hbar^2}{2m}\frac{1}{\sqrt{a}}\frac{\pi^2}{a^2}\sin\left(\frac{\pi}{a}x\right) + V_0\frac{1}{\sqrt{a}}\sin\left(\frac{\pi}{a}x\right)$

$= \underbrace{\left(\frac{\hbar^2}{2m}\frac{\pi^2}{a^2} + V_0\right)}_{E}\underbrace{\frac{1}{\sqrt{a}}\sin\left(\frac{\pi}{a}x\right)}_{\psi}$

10a. Calculate $[\hat{x}, \hat{p}^2]$.

10b. Calculate $\langle \hat{x}\hat{p} \rangle$ for the *nth* state of the quantum harmonic oscillator.

10a. $[\hat{x},\hat{p}^2]f = \hat{x}\hat{p}^2 f - \hat{p}^2\hat{x}f = x\left(-i\hbar\frac{\partial}{\partial x}\right)^2 f - \left(-i\hbar\frac{\partial}{\partial x}\right)^2 x f$

$= -\hbar^2 x\frac{\partial^2 f}{\partial x^2} + \hbar^2\frac{\partial^2}{\partial x^2}(xf) = -\hbar^2 x\frac{\partial^2 f}{\partial x^2} + \hbar^2\frac{\partial}{\partial x}\left(f + x\frac{\partial f}{\partial x}\right)$

$= \cancel{-\hbar^2 x\frac{\partial^2 f}{\partial x^2}} + \hbar^2\frac{\partial f}{\partial x} + \hbar^2\frac{\partial f}{\partial x} + \cancel{\hbar^2 x\frac{\partial^2 f}{\partial x^2}} = 2\hbar^2\frac{\partial}{\partial x}f$

Or you could say: $[\hat{x},\hat{p}^2] = \hat{x}\hat{p}\hat{p} - \hat{p}\hat{p}\hat{x} = (\hat{p}\hat{x}+i\hbar)\hat{p} - \hat{p}(\hat{x}\hat{p}-i\hbar)$ using $\hat{x}\hat{p}-\hat{p}\hat{x} = i\hbar$

$= \cancel{\hat{p}\hat{x}\hat{p}} + i\hbar\hat{p} - \cancel{\hat{p}\hat{x}\hat{p}} + i\hbar\hat{p} = 2i\hbar\hat{p} = 2\hbar^2\frac{\partial}{\partial x}$

10b. $\langle \hat{x}\hat{p} \rangle = \int_{-\infty}^{\infty} \psi_n^*\hat{x}\hat{p}\psi_n\,dx = \int_{-\infty}^{\infty} \psi_n^*\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)\,i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)\psi_n\,dx$

$= i\frac{\hbar}{2}\int_{-\infty}^{\infty} \psi_n^*(\hat{a}_+ + \hat{a}_-)(\hat{a}_+ - \hat{a}_-)\psi_n\,dx$

$= i\frac{\hbar}{2}\int_{-\infty}^{\infty} \psi_n^*(\cancelto{0}{\hat{a}_+\hat{a}_+} + \hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- - \cancelto{0}{\hat{a}_-\hat{a}_-})\psi_n\,dx$

$= i\frac{\hbar}{2}\int_{-\infty}^{\infty} \psi_n^*(n+1-n)\psi_n\,dx = i\frac{\hbar}{2}\cancelto{1}{\int_{-\infty}^{\infty}\psi_n^*\psi_n\,dx} = i\frac{\hbar}{2}$

**Grande:** *(30 pts)* Choose one of the two potentials below to solve. In either case, if the spectrum is discrete find an explicit expression for $c_n$. If it is continuous find the function $\phi(k)$.

Please indicate your steps clearly. Your answers do not have to be extravagantly long, just clear in what you are doing. **If it is a continuous case, you do not need to normalize the solution to the TISE. Just set the coefficient of $\psi(k, x)$ to 1 in this case.**

11a. "Particle Meets Wall"

Consider the potential given by $V(x) = \begin{cases} \infty & x < 0 \\ 0 & x > 0 \end{cases}$. The particle starts at $t = 0$ with the initial wavefunction $\Psi(x, 0) = \begin{cases} 0 & x < b \\ A & b < x < 2b \\ 0 & x > 2b \end{cases}$. Find $c_n$ or $\phi(k)$. **Hint: Use real solutions.**

11b. "Particle In An Extra Dimension"

The Kaluza-Klein model of extra dimensions assumes that they are small circles. Consider a free particle which lives in one of these dimensions. The potential is zero, **but** the solution to the *TISE* must satisfy periodic boundary conditions, i.e. $\psi(0) = \psi(L)$ and $\frac{d\psi}{dx}|_0 = \frac{d\psi}{dx}|_L$ where $L$ is the circumference of the circle. **Note:** There is no infinity here, you only integrate $x$ from $0$ to $L$. The particle starts off at $t = 0$ in the state $\Psi(x, 0) = A\cos(\frac{6\pi}{L}x)$. Find $c_n$ or $\phi(k)$.

**Hint: Use real solutions.**

11a. TISE $\frac{d^2\psi}{dx^2} = \underbrace{-\frac{2mE}{\hbar^2}}_{-k^2 \text{ since } E>0}\psi$ $\Rightarrow \psi = A\sin(kx) + B\cos(kx)$

At $x=0$ need $\psi$ to be zero for continuity with solution for $x<0$, i.e. $\psi(x)=0$ $x<0$.

Thus: $B=0$

$\psi(k,x) = A\sin(kx)$ not-normalizable so set $A=1$

$\psi(k,x) = \sin(kx)$

Then: $\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{-i\frac{E(k)}{\hbar}t}\psi(k,x)\,dk$

And: $\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi^*(k,x)\Psi(x,0)\,dx$

$= \frac{1}{\sqrt{2\pi}}\int_{b}^{2b}\sin(kx)A\,dx$

$\phi(k) = \frac{\cos(kb)-\cos(2kb)}{k\sqrt{2\pi b}}$

$= \frac{-A}{k\sqrt{2\pi}}\cos(kx)\Big|_b^{2b} = -\frac{1}{k\sqrt{2\pi b}}\left(\cos(2kb)-\cos(kb)\right)$

To find $A$: $\int_b^{2b}\Psi^*\Psi\,dx = 1 \Rightarrow A^2\int_b^{2b}dx = A^2 b = 1$

$A = \frac{1}{\sqrt{b}}$

**Grande:** *(30 pts)* Choose one of the two potentials below to solve. In either case, if the spectrum is discrete find an explicit expression for $c_n$. If it is continuous find the function $\phi(k)$.

Please indicate your steps clearly. Your answers do not have to be extravagantly long, just clear in what you are doing. **If it is a continuous case, you do <u>not</u> need to normalize the solution to the TISE. Just set the coefficient of $\psi(k,x)$ to 1 in this case.**

11a. "Particle Meets Wall"
Consider the potential given by $V(x) = \begin{cases} \infty & x<0 \\ 0 & x>0 \end{cases}$. The particle starts at $t=0$ with the initial wavefunction $\Psi(x,0) = \begin{cases} 0 & x<b \\ A & b<x<2b \\ 0 & x>2b \end{cases}$. Find $c_n$ or $\phi(k)$. **Hint: Use real solutions.**

11b. "Particle In An Extra Dimension"
The Kaluza-Klein model of extra dimensions assumes that they are small circles. Consider a free particle which lives in one of these dimensions. The potential is zero, **but** the solution to the *TISE* must satisfy periodic boundary conditions, i.e. $\psi(0)=\psi(L)$ and $\frac{d\psi}{dx}|_0 = \frac{d\psi}{dx}|_L$ where $L$ is the circumference of the circle. **Note:** There is no infinity here, you only integrate $x$ from $0$ to $L$. The particle starts off at $t=0$ in the state $\Psi(x,0) = A\cos(\frac{6\pi}{L}x)$. Find $c_n$ or $\phi(k)$.
**Hint: Use real solutions.**

11b. TISE $\frac{d^2\psi_n}{dx^2} = \underbrace{-\frac{2mE_n}{\hbar^2}}_{-k^2 \text{ since } E>0}\psi_n$ $\Rightarrow \psi_n(x) = A\sin(kx) + B\cos(kx)$

Assume even or odd since $V$ is symmetric

E: $\psi_n(x) = B\cos(kx)$

O: $\psi_n(x) = A\sin(kx)$

Apply: $\psi_n(0) = \psi_n(L) \Rightarrow$ E: $B = B\cos(kL)$
O: $0 = A\sin(kL)$

$\frac{d\psi_n}{dx}\Big|_0 = \frac{d\psi_n}{dx}\Big|_L \Rightarrow$ E: $0 = -Bk\sin(kL)$
O: $Ak = Ak\cos(kL)$

The boundary conditions are satisfied when:

E: $k = \frac{2n\pi}{L}$ $\Rightarrow$ $\psi_{E_n}(x) = B\cos(\frac{2n\pi}{L}x)$

O: $k = \frac{2n\pi}{L}$ $\Rightarrow$ $\psi_{O_n}(x) = A\sin(\frac{2n\pi}{L}x)$

Normalizing each: $\int_0^L \psi_{E_n}^* \psi_{E_n} dx = B^2\frac{L}{2} = 1 \Rightarrow B = \sqrt{\frac{2}{L}}$

$\int_0^L \psi_{O_n}^* \psi_{O_n} dx = A^2\frac{L}{2} = 1 \Rightarrow A = \sqrt{\frac{2}{L}}$

If we compare $\Psi(x,0)$ to the stationary solutions we immediately see that $\Psi(x,0) = \psi^E_{n=3}$ so we know that $c_3 = 1$ and $c_{n\neq 3} = 0$.