Big: *(5 pts each)*

1. Are all solutions to the T.I.S.E. probability amplitudes? No, only the normalized solutions.

2. For a stationary state of a quantum system one calculates $\langle \hat{H} \rangle = 32\, J$. What is $\langle \hat{H}^2 \rangle$ for this system? If it cannot be determined, explain why.

For stationary states $\sigma_H = 0 = \sqrt{\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2} \Rightarrow \langle \hat{H}^2 \rangle = \langle \hat{H} \rangle^2 = 32^2\, J^2$

3. For an even potential $V(x)$) we can, without loss of generality, use an orthormal set of basis functions $\psi_n(x)$ which are even (for n-even) and odd (for n-odd) under $x \rightarrow -x$. With this basis is $\int_{-\infty}^{\infty} \psi_n^* \psi_{n+2}\, dx$ necessarily equal to 0?

Orthonormality says $\int_{-\infty}^{\infty} \psi_n^* \psi_m\, dx = \delta_{nm}$ since $n \neq n+2$ then $\int_{-\infty}^{\infty} \psi_n^* \psi_{n+2}\, dx = 0$

4. For what range of k is the function $f(x) = e^{kx}$ in Hilbert space on the interval (-1,1)? **You may assume that k is real.**

$\int_{-1}^{1} f^* f\, dx < \infty \Rightarrow \int_{-1}^{1} e^{2kx}\, dx = \frac{1}{2k}\left[e^{2k} - e^{-2k}\right] < \infty \Rightarrow$ requires $|k| < \infty$

Note: $k = 0$ is allowed since $\int_{-1}^{1} e^{0} dx = 2$

5. Circle (**T**)rue or (**F**)alse for each statement below:

T or **F**: Expectation values of hermitian operators are always real.
T or **F**: Eigenfunctions of hermitian operators are always orthonormal. They are not necessarily normalized!
T or **F**: Non-extended Hilbert space contains only normalized functions. They must be normalizable.
T or **F**: Hermitian operators are always equal to their complex conjugate. Counterexample is $\hat{p} = -i\hbar \frac{\partial}{\partial x}$
T or **F**: Eigenvalues of hermitian operators are always real.

6. Circle (**T**)rue or (**F**)alse for each statement below:

T or **F**: Angular solutions to the T.I.S.E. for $V = -k(x^2 + y^2 + z^2)^{1/3}$ are the spherical harmonics. $V = -k r^{2/3}$
T or **F**: Ignoring spin, every energy level of the hydrogen atom is degenerate. $n=1$ only allows $\ell = 0$ and $\ell = 0$ only allows $m = 0$
T or **F**: The hydrogen (ψ_{nlm}) state ψ_{432} is higher in energy than ψ_{421}. E only depends on n.
T or **F**: The largest value of m occurs when the angular momentum vector is perfectly aligned along the positive z-axis, that is, when the eigenvalues satusfy $L_z = \sqrt{L^2}$. $L_{z\,max} = \ell\hbar$, $\sqrt{L^2} = \hbar\sqrt{\ell^2 + \ell} \neq L_{z\,max}$
T or **F**: A spin-3/2 particle would have 3 possible values of m. $\left(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right)$

7. A beam of spin-1/2 particles with randomly oriented spins is sent through a Stern-Gerlach device setup along the z-axis. Then all of the particles that are deflected along $+z$ are collected and sent through a second SG device along the x-axis. The particles that are deflected along $+x$ are collected and then sent through a third SG device again along the z-axis. What percentage of the particles **going into the third device** will deflect along the $+z$ direction?

The particles coming out of the 2nd SG are in an eigenstate of S_x. But an eigenstate of S_x is always 50% ↑ + 50% ↓ for S_z. So 50% deflect along $+z$.

8. The "Hoperator" $\hat{\hbar}$ is a hermitian operator whose action causes one electron to play leap frog over another. Given the operator equation $\hat{\hbar}\, g = af$ for two normalized states f and g, calculate the quantity $\langle g | \hat{\hbar}\, f \rangle$. **Be careful about making assumptions in this problem.**

$$\langle g | \hat{\hbar} f \rangle = \langle \hat{\hbar} g | f \rangle = \langle af | f \rangle = a^* \underbrace{\langle f | f \rangle}_{1 \text{ normalized}} = \underbrace{a^*}$$

Note: $\hat{\hbar} g = af$ is not an eigenvalue eqn $\Rightarrow$ cannot assume that a is real!

Bigger: *(15 pts)* Choose one of the questions below (9a or 9b) to answer.

9a. Starting from the definition of hermiticity, $\langle \hat{Q}f|g\rangle = \langle f|\hat{Q}g\rangle$, prove that $\hat{p}$ is a hermitian operator. **Hint: Use the integral form of the inner product and assume f and g are functions of x.**

9b. Calculate the reflection coefficient for a particle incident from the left with $E < V_0$ on the step potential $(x) = \begin{cases} 0, & x \le 0 \\ V_0, & x > 0 \end{cases}$.

from integrating by parts

9a. $\langle f|\hat{p}g\rangle = \int_{-\infty}^{\infty} f^*(x)(-i\hbar\frac{\partial}{\partial x})g(x)\,dx = \int_{-\infty}^{\infty}\left[-i\hbar\frac{\partial}{\partial x}(f^*g) + i\hbar\frac{\partial f^*}{\partial x}g\right]dx$

$= \cancel{-i\hbar f^* g\Big|_{-\infty}^{\infty}} + \int_{-\infty}^{\infty}\left(-i\hbar\frac{\partial f}{\partial x}\right)^* g\,dx$

assume 0

$\boxed{\langle f|\hat{p}g\rangle = \langle\hat{p}f|g\rangle}$ therefore $\hat{p}$ is hermitian.

Bigger part deux: *(15 pts)* Choose one of the questions below (10a or 10b) to answer.

10a. Find the minimum value of the product of the standard deviations in measurements of S_x and S_z for a large collection of spin-1/2 particles in the state χ_+^y.

10b. Find the normalized eigenspinors of $\hat{S}_y$. **You may assume the eigenvalues are known.**

10a. $\sigma_{S_x}^2\sigma_{S_z}^2 \geq \left(\frac{1}{2i}\langle[\hat{S}_x,\hat{S}_z]\rangle_{\chi_+^y}\right)^2 = \left(\frac{1}{2i}\langle -i\hbar\hat{S}_y\rangle_{\chi_+^y}\right)^2 = \left(\frac{-\hbar}{2}\langle\chi_+^y|\hat{S}_y\chi_+^y\rangle\right)^2$

$= \left(-\frac{\hbar}{2}\left(+\frac{\hbar}{2}\right)\langle\chi_+^y|\chi_+^y\rangle^{\nearrow 1}\right)^2$

$= \frac{\hbar^4}{4^2}$

So $\boxed{\sigma_{S_x}\sigma_{S_z\,min} = \frac{\hbar^2}{4}}$

You do not need χ_+^y to evaluate this! You only need the eigenvalue of $\hat{S}_y$ which is $+\frac{\hbar}{2}$.

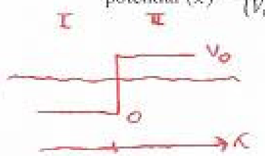
Bigger:(*15 pts*) Choose one of the questions below (9a or 9b) to answer.

9a. Starting from the definition of hermiticity, $\langle \hat{Q} f | g\rangle = \langle f | \hat{Q} g\rangle$, prove that $\hat{p}$ is a hermitian operator. **Hint: Use the integral form of the inner product and assume f and g are functions of x.**

9b. Calculate the reflection coefficient for a particle incident from the left with $E < V_o$ on the step potential $(x) = \begin{cases} 0, & x \le 0 \\ V_{0,} & x > 0 \end{cases}$

9b.

$0 < E < V_0$



I: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi$ $\frac{\partial^2\psi}{\partial x^2} = -\frac{2mE}{\hbar^2}\psi = -k_I^2\psi$

II: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V_0\psi = E\psi$ $\frac{\partial^2\psi}{\partial x^2} = -\frac{2m(E-V_0)}{\hbar^2}\psi = k_{II}^2\psi$

I: $\psi = Ae^{ik_I x} + Be^{-ik_I x}$

II: $\psi = Fe^{k_{II} x} + Ge^{-k_{II} x}$

For scattering from left keep A, B but set F = 0 since it blows up at $+\infty$.

Continuity at x = 0: $A + B = G$ Smoothness at x = 0: $ik_I A - ik_I B = -k_{II} G$

$R = \frac{\text{prob. of refl}}{\text{prob. of inc.}} = \frac{\int |\psi_R|^2 dx}{\int |\psi_I|^2 dx} = \frac{|B|^2}{|A|^2} = \frac{|ik_I + k_{II}|^2}{|ik_I - k_{II}|^2}$

$\boxed{R = 1}$

Solving:
$ik_I A - ik_I B = -k_{II}(A+B)$

$(ik_I + k_{II})\frac{A}{B} = ik_I - k_{II}$

$\frac{A}{B} = \frac{ik_I - k_{II}}{ik_I + k_{II}}$

Bigger part deux:(*15 pts*) Choose one of the questions below (10a or 10b) to answer.

10a. Find the minimum value of the product of the standard deviations in measurements of S_x and S_z for a large collection of spin-1/2 particles in the state χ_+^y.

10b. Find the normalized eigenspinors of $\hat{S}_y$. **You may assume the eigenvalues are known.**

10b. $\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\hat{S}_y \chi^y_\pm = \pm\frac{\hbar}{2}\chi^y_\pm$

$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} -ib \\ ia \end{pmatrix} = \pm\begin{pmatrix} a \\ b \end{pmatrix}$

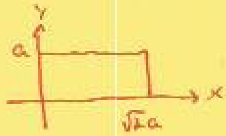
$+$: $-ib = a \Rightarrow a = 1, b = i$

$-$: $-ib = -a \Rightarrow a = 1, b = -i$

$\Rightarrow \chi^y_\pm = \begin{pmatrix} 1 \\ \pm i \end{pmatrix} A$

Normalizing: $(\chi^y_\pm)^\dagger(\chi^y_\pm) = 1 = A^2(1 \ \mp i)\begin{pmatrix} 1 \\ \pm i \end{pmatrix} = 2A^2 \Rightarrow A = \frac{1}{\sqrt{2}}$

$\boxed{\chi^y_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}}$ Note: other possibilities exist!

11a. 

i) $-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)=E\psi \Rightarrow \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} = -\frac{2mE}{\hbar^2}\psi$

Assume: $\psi(x,y) = X(x)\,Y(y)$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = -\frac{2mE}{\hbar^2}XY$$

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{=-k_x^2} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{=-k_y^2} = -\frac{2mE}{\hbar^2}$$

Then: $\frac{\partial^2 X}{\partial x^2} = -k_x^2 X \Rightarrow X(x) = A\cos(k_x x)+B\sin(k_x x)$ For $X(0)=0 \Rightarrow A=0$

$Y(y) = F\cos(k_y y)+G\sin(k_y y)$ For $Y(0)=0 \Rightarrow F=0$

$X(x) = B\sin(k_x x)$ For $X(\sqrt{2}a)=0 \Rightarrow k_x = \frac{n_x\pi}{\sqrt{2}a}$ $n_x = 1,2,\dots$

$Y(y) = G\sin(k_y y)$ For $Y(a)=0 \Rightarrow k_y = \frac{n_y\pi}{a}$ $n_y=1,2,\dots$

Then: $\psi(x,y) = A\sin\left(\frac{n_x\pi}{\sqrt{2}a}x\right)\sin\left(\frac{n_y\pi}{a}y\right)$

$$\int_0^a\int_0^{\sqrt{2}a}\psi^*\psi\,dx\,dy = 1 = A^2\int_0^{\sqrt{2}a}\sin^2\left(\frac{n_x\pi}{\sqrt{2}a}x\right)dx\int_0^a\sin^2\left(\frac{n_y\pi}{a}y\right)dy = A^2\left(\frac{\sqrt{2}a}{2}\right)\left(\frac{a}{2}\right) = A^2\frac{\sqrt{2}}{4}a^2$$

$A = \frac{2a}{2^{1/4}}$

So: $\boxed{\psi(x,y) = \frac{2a}{2^{1/4}}\sin\left(\frac{n_x\pi}{\sqrt{2}a}x\right)\sin\left(\frac{n_y\pi}{a}y\right)}$

$E_{n_x n_y} = \frac{\hbar^2}{2m}(k_x^2+k_y^2) = \frac{\hbar^2}{2m}\frac{\pi^2}{a^2}\left(\frac{n_x^2}{2}+n_y^2\right) = \boxed{\frac{\hbar^2}{4m}\frac{\pi^2}{a^2}\left(n_x^2+2n_y^2\right)}$

ii)

n_x	n_y	$E/\frac{\hbar^2}{4m}\frac{\pi^2}{a^2}$	deg.
1	1	3	1
2	1	6	1
3	1	11	1
1	2	9	1
2	2	12	1

cont.

n_x	n_y	$E/$	deg.
3	2	17	1
4	1	18	1

$\langle\hat{H}\rangle = \frac{1}{2m}\langle\hat{p}^2\rangle$

iii) $\hat{H} = \frac{1}{2m}(\hat{p}_x^2+\hat{p}_y^2)$

$\hat{H}\psi_{11} = 3\frac{\hbar^2}{4m}\frac{\pi^2}{a^2}\psi_{11}$

$\langle\hat{H}\rangle = 3\frac{\hbar^2}{4m}\frac{\pi^2}{a^2}$

$\langle\hat{p}_x^2+\hat{p}_y^2\rangle = 2m\langle\hat{H}\rangle = \boxed{\frac{3\hbar^2}{2}\frac{\pi^2}{a^2}}$

6. i) $-\frac{\hbar^2}{2m}\nabla^2\psi + V(r)\psi = E\psi$

$$-\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right) + V(r)\psi = E\psi$$

Assume $\psi = R(r)Y(\theta,\phi)$:

$$Y\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + R\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + R\frac{1}{r^2\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -\frac{2m}{\hbar^2}(E-V(r))RY$$

Multiply through by $\frac{r^2}{RY}$:

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{Y}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{Y}\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -\frac{2m}{\hbar^2}(E-V(r))r^2$$

Move r to left and θ,ϕ to right:

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{2m}{\hbar^2}(E-V(r))r^2 = -\frac{1}{Y}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{Y}\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}$$

Each side can vary independently so must be a constant. Call it $l(l+1)$:

$$\begin{array}{ll}\text{Radial:} & \boxed{\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{2m}{\hbar^2}(E-V(r))r^2 = l(l+1)} \\ \text{Angular:} & \boxed{-\frac{1}{Y}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{Y}\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = l(l+1)}\end{array}$$

ii) Using $R(r) = \frac{k}{r}$ and $l=1$ w/ $E=\mathcal{E}$ in the radial equation:

$$\frac{r}{k}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{k}{r}\right)\right) + \frac{2m}{\hbar^2}(\mathcal{E}-V(r))r^2 = 2$$

$$\frac{r}{k}\frac{\partial}{\partial r}\overset{0}{\cancel{(k)}} + \frac{2m}{\hbar^2}(\mathcal{E}-V(r))r^2 = 2$$

$$\boxed{V(r) = \mathcal{E} - \frac{\hbar^2}{mr^2}}$$

iii) For any central potential, the solutions to the angular part are the spherical harmonics $Y_l^m(\theta,\phi)$.

We know that for $l=1$ there are always 3 possible values of m $(-1,0,1)$. Since the energy does not depend on m, this means all 3 values have the same energy so the degeneracy is $\boxed{3}$.