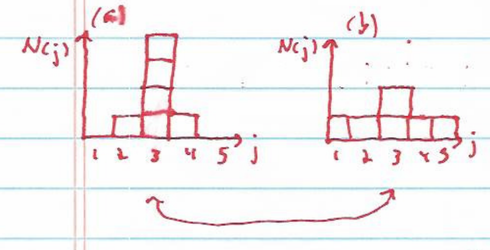
Review

a) Prob. dist. $\frac{N(j)}{N}$ or $\underset{\text{density}}{\rho(x)}dx$ must be normalized! $\sum_j \frac{N(j)}{N} = 1$, $\int_{-\infty}^{\infty} \rho(x)dx = 1$

b) [illegible] averages (exp. values): $\langle f(j)\rangle = \sum_j f(j)\frac{N(j)}{N}$ $\langle f(x)\rangle = \int_{-\infty}^{\infty} f(x)\rho(x)dx$

Snowball Slob!

Consider:



$\underbrace{\text{Same \# elements (6), average } (\tfrac{18}{6} = 3)\text{, median (3), most prob. (3)}}_{\text{desc. in terms of these prob. quantities is incomplete}}$

But obviously different!

To help (not complete) distinguish: $\sigma = \sqrt{\langle(\Delta j)^2\rangle} = \sqrt{\langle j^2\rangle - \langle j\rangle^2}$ standard deviation (σ^2 is variance)

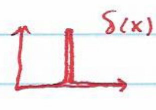
$\Delta j \equiv j - \langle j\rangle$

not at all obvious

Example above: $\left.\begin{array}{l} \sigma_a = \sqrt{\frac{56}{6} - 9} = .577 \\ \sigma_b = \sqrt{\frac{64}{6} - 9} = 1.29 \end{array}\right\}$ measures "width" of dist.

For $\rho(x)$: $\sigma = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$

For $\sigma = 0 \Rightarrow \langle x^2\rangle = \langle x\rangle^2 \Rightarrow$



Quantum Mechanics

time-dep. SE $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$ Controls dynamical evolution of $\Psi(x,t)$

$\uparrow$ pot. energy function $V(x,t)$

Solve SE for $\Psi(x,t)$ - ~~prob. amplitude~~ wave function

What is it? $\Psi(x,t)$ is a prob. amplitude $\Rightarrow$ $|\Psi(x,t)|^2 = \Psi^*\Psi$ = prob. density

(like $\rho(x)$)

So: $\int_a^b |\Psi|^2 dx$ prob. of finding particle btwn a and b

$\langle x \rangle = \int_{-\infty}^{\infty} x|\Psi|^2 dx$ exp. value of position (average value of x)

What about t? (always $\int dx$) For NRQM we do measurements at fixed t

so $\Psi(x,t)$ determines prob. of outcomes at t.

Wait a minute: Is there any guarantee $\int_{-\infty}^{\infty} |\Psi|^2 dx = 1$? (req. for prob. dist. int.)

2 Issues a) NO! Solving SE for Ψ does not automatically give $\int |\Psi|^2 = 1$

but for any Ψ, $A\Psi$ is also a solution and we can

be careful to always choose the one such that $\int |\Psi|^2 = 1$.

Choosing the right A is called normalizing the wavefunction.

It is required!!

b) But also note $\underbrace{\int_{-\infty}^{\infty} \Psi^*(x,t)\,\Psi(x,t)\,dx}_{f(t)} = 1$ $\leftarrow$ no t-dep.

If we choose A s.t. $\int |\Psi|^2 dx = 1$ for some t, could this change?

$\Psi(x,t)$ evolves by TDSE and it says no.

$$\frac{d}{dt}\int_{-\infty}^{\infty} \psi^* \psi \, dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}(\psi^* \psi)\, dx$$

$$= \int_{-\infty}^{\infty} \left[\frac{\partial \psi^*}{\partial t} \psi + \psi^* \frac{\partial \psi}{\partial t} \right] dx$$

SE $\quad \frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m} \frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} V\psi \quad$ or $\quad \frac{\partial \psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + \frac{i}{\hbar} V \psi^*$

$$= \int_{-\infty}^{\infty} \left[-\frac{i\hbar}{2m} \frac{\partial^2 \psi^*}{\partial x^2} \psi + \cancel{\frac{i}{\hbar} V\psi^* \psi} + \psi^* \frac{i\hbar}{2m} \frac{\partial^2 \psi}{\partial x^2} - \cancel{\frac{i}{\hbar} V \psi^* \psi} \right] dx$$

$$= \frac{i\hbar}{2m} \int_{-\infty}^{\infty} \left[-\frac{\partial^2 \psi^*}{\partial x^2} \psi + \psi^* \frac{\partial^2 \psi}{\partial x^2} \right] dx$$

$$= \frac{-i\hbar}{2m} \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial x}\left[\frac{\partial \psi^*}{\partial x} \psi \right] - \frac{\partial \psi^*}{\partial x}\frac{\partial \psi}{\partial x} \right\} dx$$

$$+ \frac{i\hbar}{2m} \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial x}\left[\psi^* \frac{\partial \psi}{\partial x} \right] - \frac{\partial \psi^*}{\partial x}\frac{\partial \psi}{\partial x} \right\} dx$$

$$= \frac{i\hbar}{2m} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left[-\frac{\partial \psi^*}{\partial x} \psi + \psi^* \frac{\partial \psi}{\partial x} \right] dx$$

$$= \frac{-i\hbar}{2m} \left[\frac{\partial \psi^*}{\partial x} \psi + \psi^* \frac{\partial \psi}{\partial x} \right]_{-\infty}^{\infty}$$

Assuming $\psi(x,t)\Big|_{\pm\infty} = 0$ (true for normalizable wavefunctions anyway)

$\frac{d}{dt}\int_{-\infty}^{\infty} |\psi|^2 dx = 0 \Rightarrow \int_{-\infty}^{\infty} |\psi|^2 dx$ ind. of time

TDSE preserves normalization of ψ!

Momentum

CM: Solve e.o.m. $\Rightarrow \vec{x}(t)$ then use $\vec{p} = m\dot{\vec{x}}$

QM: We don't have $\vec{x}(t)$ (we have $\Psi(x,t)$).

What do we have? $\langle x \rangle$ is closest

So let's define: $\langle p \rangle = m \underbrace{\frac{d\langle x \rangle}{dt}}$ (or $\langle v \rangle = \frac{d\langle x \rangle}{dt}$)

Note: $\frac{d}{dt}\int |\Psi|^2 dx = 0$

but $\frac{d}{dt}\int x|\Psi|^2 dx$ does not

Massaging: $\langle p \rangle = m\frac{d\langle x \rangle}{dt} = m\frac{d}{dt}\int_{-\infty}^{\infty} x\Psi^*\Psi\, dx$

$= m\int_{-\infty}^{\infty} x \frac{\partial}{\partial t}\left[\Psi^*(x,t)\Psi(x,t)\right]dx$

using TDSE again $= \frac{i\hbar}{2}\int_{-\infty}^{\infty} \underbrace{x\frac{\partial}{\partial x}\left[\Psi^*\frac{\partial\Psi}{\partial x}\right.}_{\frac{\partial}{\partial x}\left[x\Psi^*\frac{\partial \Psi}{\partial x}\right] - \Psi^*\frac{\partial\Psi}{\partial x}} \left. - \frac{\partial \Psi^*}{\partial x}\Psi\right]dx$

$\underbrace{\frac{\partial}{\partial x}\left[x\Psi^*\frac{\partial \Psi}{\partial x}\right]}$

boundary term $\to 0$ for $x \to \pm\infty$

$= -\frac{i\hbar}{2}\int_{-\infty}^{\infty}\left[\Psi^*\frac{\partial\Psi}{\partial x} - \underbrace{\frac{\partial\Psi^*}{\partial x}\Psi}_{\frac{\partial}{\partial x}\left[\Psi^*\Psi\right] - \Psi^*\frac{\partial\Psi}{\partial x}}\right]dx$

$= -i\hbar\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}dx$ boundary term $\to 0$

$= \int_{-\infty}^{\infty}\Psi^*\underbrace{(-i\hbar\partial_x)}\Psi\, dx$

suggest $p = -i\hbar\partial_x$

In general: $\langle f(x,p) \rangle = \int_{-\infty}^{\infty}\Psi^*\underbrace{f(x,p)}_{\text{ordering within will require care}}\Psi dx$ Then $\langle p \rangle = \underbrace{\int_{-\infty}^{\infty}\Psi^* p \Psi}\, dx$

Must be careful of ordering !!

Also $\langle x \rangle = \int_{-\infty}^{\infty}\Psi^* x\Psi\, dx$

Uncertainty

Now that we can compute $\langle p \rangle$ we could also do $\langle p^2 \rangle$ and find:

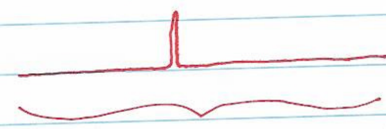
$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

Of course $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

Then: $\sigma_x \sigma_p \geq \frac{\hbar}{2}$

Much more on this later (including derivation) but for now:

To squeeze $\sigma_x \to 0$



Fourier expanding will require all ~~frequency~~ wavelength modes $\lambda_n \neq 0$
but $p_n = \frac{2\pi\hbar}{\lambda_n}$ so all p_n values are included.
Definitely not a pure momentum state.

To squeeze $\sigma_p \to 0$

λ or p has one definite value

Clearly no well defined "position" of wave
In fact it must be infinitely long to
be perfectly single λ!