Review

TDSE: $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi$

$\langle \hat{f} \rangle$ - is ind. of t

$\psi_n(x,t)$ "stationary state"

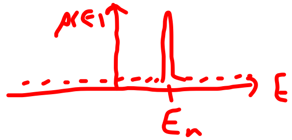
Assume $\Psi(x,t) = \sum_n c_n \, \phi_n(t) \psi_n(x)$

Then: TDSE $\Rightarrow$ $\phi_n(t) = e^{-i\frac{E_n}{\hbar}t}$ (normalized)

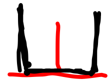
$\Rightarrow$ $\boxed{\hat{H}\psi_n(x) = E_n \psi_n(x)}$ TISE

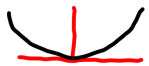
$\underbrace{\hat{H}}_{-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)}$ Hamiltonian

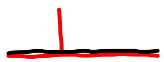
$\rho(E) = \delta(E - E_n)$



We will solve TISE for various $V(x)$:

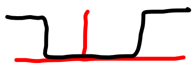
Infinite well  (simplest)

Harmonic osc.  (most important)

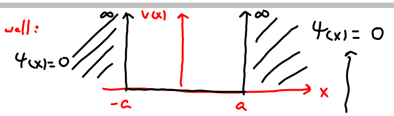
Zero (constant)  (non-normalizable ψ's)

Combine scattering & Bound states $\left\{ \begin{array}{l} \delta\text{-function} \\ \text{Finite well} \end{array} \right.$

δ-function 

Finite well 

Infinite well:



not true for finite well

$$V(x) = \begin{cases} 0 & |x| \le a \\ \infty & |x| > a \end{cases}$$

TISE $\quad -\frac{\hbar^2}{2m}\frac{d^2\psi_n(x)}{dx^2} + V(x)\psi_n(x) = E_n\psi_n(x)$

$|x| > a \quad V(x) = \infty \quad \underbrace{\frac{d^2\psi}{dx^2} = \infty\psi}_{\text{non-normalizable}} \quad$ HW 2.2 $\quad (\psi = 0)$

$$|x| < a \quad \frac{d^2\psi_n(x)}{dx^2} = -\underbrace{\frac{2mE_n}{\hbar^2}}_{k^2}\psi_n(x)$$

$$\psi_n(x) = A\cos(kx) + B\sin(kx)$$

Use spatial boundary conditions:

Continuity: $\psi(-a) = \psi(a) = 0$ matches $\psi(x)$ $|x| > a$

~~Smoothness: $\frac{\partial\psi}{\partial x}$ continuous~~

allowed for singular potential

$$\psi_n(a) = A\cos(ka) + B\sin(ka) = 0$$

2 ways to solve: $\left.\begin{array}{ll} A=0 & k=\frac{n\pi}{2a} \quad n=\text{even} \\ B=0 & k=\frac{n\pi}{2a} \quad n=\text{odd} \end{array}\right\}$ Automatically satisfies $\psi_n(-a)=0$

Normalizing:

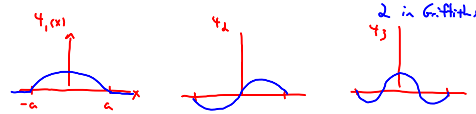
$\psi_n(x) = A\cos(kx) \Rightarrow A^2\int_{-a}^{a}\cos^2\left(\frac{n\pi}{2a}x\right)dx = A^2 a = 1 \Rightarrow A = \frac{1}{\sqrt{a}}$

$\psi_n(x) = B\sin(kx) \Rightarrow B^2\int_{-a}^{a}\sin^2\left(\frac{n\pi}{2a}x\right)dx = B^2 a = 1 \Rightarrow B = \frac{1}{\sqrt{a}}$

Solutions: $\psi_n(x) = \begin{cases} \frac{1}{\sqrt{a}}\cos\left(\frac{n\pi}{2a}x\right) & n=\text{odd} \\ \frac{1}{\sqrt{a}}\sin\left(\frac{n\pi}{2a}x\right) & n=\text{even} \end{cases}$

Energy levels (E_n)

Recall: $k^2 = \frac{2mE_n}{\hbar^2} \Rightarrow E_n = \frac{\hbar^2k^2}{2m} = \frac{\hbar^2\pi^2}{\underbrace{8}_{2\text{ in Griffiths}}ma^2}n^2$



$n=1$ cos $\qquad$ $n=2$ sin $\qquad$ $n=3$ cos

$\underbrace{\psi_1 - \text{even} \qquad \psi_2 - \text{odd} \qquad \psi_3 - \text{even}}_{\text{If } V(-x) = \pm V(x)}$ $\quad$ HW 2.1

Finally TDSE:

$$\Psi_{(x,t)} = \sum_n c_n \phi_n(t) \psi_n(x)$$

$$= \sum_{n\text{-odd}} c_n e^{-i\frac{\hbar\pi^2}{8ma^2}n^2 t} \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi}{2a}x\right)$$

$$+ \sum_{n\text{-even}} c_n e^{-i\frac{\hbar\pi^2}{8ma^2}n^2 t} \frac{1}{\sqrt{a}} \cos\left(\frac{n\pi}{2a}x\right)$$

I promised: $\Psi_{(x,t)} = \sum_n c_n \underbrace{\psi_n(x,t)}_{\text{orthonormal basis set}}$

$\psi_n(x,t) = \phi_n(t)\psi_n(x)$

Orthogonality: $\int_{-a}^{a} \psi_m^*(x,t)\psi_n(x,t)\,dx \overset{?}{=} 0 \quad m \neq n$

$$\int_{-a}^{a} \psi_m^*(x)\psi_n(x)\,dx$$

Use: $\int_{-a}^{a} \sin\left(\frac{m\pi}{2a}x\right)\sin\left(\frac{n\pi}{2a}x\right)dx = 0$ if $m \neq n$

$\int_{-a}^{a} \sin\left(\frac{m\pi}{2a}x\right)\cos\left(\frac{n\pi}{2a}x\right)dx = 0$ trivially

$\int_{-a}^{a} \cos\left(\frac{m\pi}{2a}x\right)\cos\left(\frac{n\pi}{2a}x\right)dx = 0$ if $m \neq n$

$\int_{-a}^{a} \psi_m^*(x)\psi_n(x)\,dx = 0 \quad m \neq n$

Normality

$$\int_{-a}^{a} \psi_n^*(x,t)\psi_n(x,t)\,dx \overset{?}{=} 1$$

$$\int_{-a}^{a} \psi_n^2(x)\,dx = 1$$

Generally: $\int_{-a}^{a} \psi_m^*(x,t)\psi_n(x,t)\,dx = \delta_{mn} = \begin{cases} 1 & m=n \\ 0 & m \neq n \end{cases}$

For $\Psi_{(x,t)}$: $\int_{-\infty}^{\infty} \psi_1^*(x,t)\Psi_{(x,t)}\,dx = c_1$

overlap of Ψ w/ ψ_1

What are c_n? $|c_n|^2$ = probability of measuring E_n for $\Psi_{(x,t)}$

Note: $\int_{-a}^{a} \Psi^*_{(x,0)} \overset{\phi=1}{\Psi_{(x,0)}}\,dx = \int_{-a}^{a} \left(\sum_n c_n \psi_n(x)\right)^* \left(\sum_m c_m \psi_m(x)\right) dx$

$$= \sum_n \sum_m c_n^* c_m \underbrace{\int_{-a}^{a} \psi_n^*(x)\psi_m(x)\,dx}_{\delta_{nm}}$$

$$= \sum_n |c_n|^2 = 1$$

$\langle \hat{H} \rangle = E_{avg} = \sum_n |c_n|^2 E_n$