Review: TDSE $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi$

Assume: $\Psi(x,t) = \sum_n c_n \underbrace{\psi_n(x,t)}_{\text{orthonormal basis functions}}$

$\psi_n(x,t) = \phi_n(t)\psi_n(x)$

$\phi_n(t)$: $e^{-i\frac{E_n}{\hbar}t}$; $\psi_n(x)$: TISE

$\underbrace{\psi_n(x),\ \psi_n(x,t),\ \Psi(x,t)}_{\text{all are normalized}}$

Harmonic Oscillator

$F = -kx \qquad \omega = \sqrt{\frac{k}{m}}$

$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$

$\psi_n(x)$

TISE: $\underbrace{-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2} + \left(\frac{1}{2}m\omega^2x^2\right)\psi_n}_{\hat{H}\psi_n} = E_n\psi_n$

$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$

Solving TISE $\Rightarrow \boxed{\psi_n(x),\ E_n}$

$$\hat{H} = \frac{1}{2m}\left[\hat{p}^2 + (m\omega\hat{x})^2\right]$$

An "inspired" construction:

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} - i\hat{p}) \qquad \hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} + i\hat{p})$$

These "almost" factorize $\hat{H}$:

$$\hat{a}_-\hat{a}_+ = \frac{1}{2\hbar m\omega}(m\omega\hat{x} + i\hat{p})(m\omega\hat{x} - i\hat{p})$$

$$= \frac{1}{2\hbar m\omega}\left(m^2\omega^2\hat{x}^2 + \hat{p}^2 - im\omega\hat{x}\hat{p} + im\omega\hat{p}\hat{x}\right)$$

$$\hat{a}_-\hat{a}_+ = \frac{1}{\hbar\omega}\hat{H} - \frac{i}{2\hbar}\underbrace{[\hat{x},\hat{p}]}_{\text{commutator of } \hat{x} \text{ and } \hat{p}}$$

$$[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} \Rightarrow [\hat{p},\hat{x}] = -[\hat{x},\hat{p}]$$

Turning inside out:

$$\hat{H} = \hbar\omega\left(\hat{a}_-\hat{a}_+ + \frac{i}{2\hbar}[\hat{x},\hat{p}]\right)$$

$$[\hat{x},\hat{p}]f(x) = (\hat{x}\hat{p} - \hat{p}\hat{x})f(x)$$

$$= \left(-i\hbar x\frac{\partial}{\partial x} + i\hbar\frac{\partial}{\partial x}x\right)f(x)$$

$$= \cancel{-i\hbar x\frac{\partial f}{\partial x}} + i\hbar f(x) + \cancel{i\hbar x\frac{\partial f}{\partial x}}$$

$$= i\hbar f(x)$$

$$\boxed{[\hat{x},\hat{p}] = i\hbar} \Rightarrow [\hat{p},\hat{x}] = -i\hbar$$

$$\hat{x}\hat{p} = \hat{p}\hat{x} + i\hbar$$

$$[\hat{a}_-,\hat{a}_+] = \hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_-$$

$$\hat{a}_-\hat{a}_+ = \frac{1}{\hbar\omega}\hat{H} - \frac{i}{2\hbar}[\hat{x},\hat{p}] \qquad \hat{a}_+\hat{a}_- = \frac{1}{\hbar\omega}\hat{H} + \frac{i}{2\hbar}[\hat{x},\hat{p}]$$

$$= -\frac{i}{\hbar}[\hat{x},\hat{p}]$$

$$[\hat{a}_-,\hat{a}_+] = 1$$

$$\hat{H} = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)$$