Review: Classical SHO $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$

$$\dot{x} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial x}$$

Quantum SHO $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 x^2$

TISE $\hat{H}\psi_n = E_n \psi_n$

$$\hat{H} = \hbar\omega\left(\hat{a}_-\hat{a}_+ + \frac{i}{2\hbar}[\hat{x},\hat{p}]\right) = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)$$

$[\hat{x},\hat{p}] = i\hbar$

$\hat{a}_-\hat{a}_+ = \hat{a}_+\hat{a}_- + 1$

$$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} \mp i\hat{p}) \Rightarrow [\hat{a}_-,\hat{a}_+] = 1$$

$$\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)$$

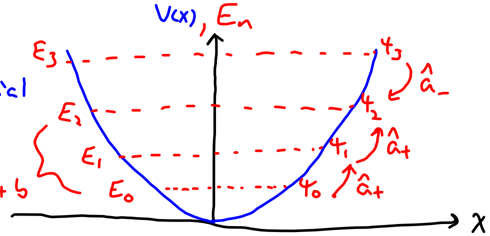
$$\Downarrow$$

$$\hat{H}\psi_n = E_n\psi_n$$

1D potential

$E \propto A^2$

$E_n = E_{n-1} + b$



Our job is to find normalized ψ_n and E_n.

$\hat{a}_+$ - raising
$\hat{a}_-$ - lowering
} ladder operators (creation, annihilation)

useful in angular momentum and in QFT

Claim: All we need to do is find ψ_0 and E_0
Everything else will come from the algebra of $\hat{a}_+, \hat{a}_-$.

Start by proving the "stepping":

Assume: $\hat{H}\psi_n = E_n\psi_n$

Consider: $\hat{H}(\hat{a}_+\psi_n) = E_{n+1}(\hat{a}_+\psi_n)$

$(a_+a_- + \frac{1}{2})a_+$

$a_+a_-a_+ + \frac{1}{2}a_+$

$a_+(a_-a_+ + \frac{1}{2})$

$$= \hbar\omega(\hat{a}_+\hat{a}_- + \tfrac{1}{2})\hat{a}_+\psi_n$$

$$= \hat{a}_+\hbar\omega(\underbrace{\hat{a}_-\hat{a}_+}_{\hat{a}_+\hat{a}_- + 1} + \tfrac{1}{2})\psi_n$$

$$= \hat{a}_+\underbrace{\hbar\omega(\hat{a}_+\hat{a}_- + \tfrac{1}{2}}_{\hat{H}} + 1)\psi_n$$

$$= \hat{a}_+(E_n + \hbar\omega)\psi_n$$

$$\hat{H}(\overbrace{\hat{a}_+\psi_n}^{\psi_{n+1}}) = (\overbrace{E_n + \hbar\omega}^{E_{n+1}})(\hat{a}_+\psi_n)$$

is a solution to TISE

You derive: $\hat{H}(\hat{a}_-\psi_n) = (E_n - \hbar\omega)(\hat{a}_-\psi_n)$

$$\boxed{\hat{H}\psi_{n+1} = E_{n+1}\psi_{n+1} \qquad \begin{array}{l}\psi_{n+1} \propto \hat{a}_+\psi_n \\ \psi_{n-1} \propto \hat{a}_-\psi_n\end{array}}$$

<u>Caution</u>: $\psi_{n+1} = \hat{a}_+ \psi_n$ is a solution to TISE but not nec. normalized

$-i\hat{p} \uparrow -i\hbar \frac{d}{dx}$

$$\int_{-\infty}^{\infty} \psi_{n+1}^* \psi_{n+1}\, dx = 1$$

$$= \int_{-\infty}^{\infty} (\hat{a}_+ \psi_n)^* (\hat{a}_+ \psi_n)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2m\hbar\omega}} \left((m\omega \hat{x} - i\hat{p}) \psi_n \right)^* \hat{a}_+ \psi_n\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2m\hbar\omega}} \left(m\omega x \psi_n - \hbar \frac{d\psi_n}{dx} \right)^* \hat{a}_+ \psi_n\, dx$$

We know $x\psi = \psi x$

But $\frac{d}{dx}$ is tricky: $\int \left(\frac{d}{dx}\psi\right)(\text{crap})\, dx$

$$\frac{d}{dx}(\psi\,\text{crap}) - \psi \frac{d\,\text{crap}}{dx}$$

$$\int_{-\infty}^{\infty} \frac{d}{dx}(\psi\,\text{crap})\, dx - \int \psi \frac{d\,\text{crap}}{dx}\, dx$$

$$\psi\,\text{crap}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \psi \frac{d\,\text{crap}}{dx}\, dx$$

0

$$= \int_{-\infty}^{\infty} \psi_n^* \underbrace{\frac{1}{\sqrt{2m\hbar\omega}} \left(m\omega x + \hbar \frac{d}{dx} \right)}_{\hat{a}_-} \hat{a}_+ \psi_n\, dx$$

$$= \int_{-\infty}^{\infty} \psi_n^* \underbrace{\hat{a}_- \hat{a}_+}_{\frac{\hat{H}}{\hbar\omega} + \frac{1}{2}} \psi_n\, dx$$

$$= \int_{-\infty}^{\infty} \psi_n^* \left(\frac{\hat{H}}{\hbar\omega} + \frac{1}{2} \right) \psi_n\, dx$$

$$= \int_{-\infty}^{\infty} \psi_n^* \left(\frac{E_n}{\hbar\omega} + \frac{1}{2} \right) \psi_n\, dx$$

$$= \left(\frac{E_n}{\hbar\omega} + \frac{1}{2} \right) \int_{-\infty}^{\infty} \psi_n^* \psi_n\, dx \quad 1$$

$$\int_{-\infty}^{\infty} (\hat{a}_+ \psi_n)^* (\hat{a}_+ \psi_n)\, dx = \frac{E_n}{\hbar\omega} + \frac{1}{2}$$

$$\boxed{\psi_{n+1} = \frac{1}{\sqrt{\frac{E_n}{\hbar\omega} + \frac{1}{2}}} \hat{a}_+ \psi_n}$$

↑ gives a normalized solution to TISE <u>iff</u> ψ_n is a norm. sol. to TISE

You prove: $\boxed{\psi_{n-1} = \frac{1}{\sqrt{\frac{E_n}{\hbar\omega} - \frac{1}{2}}} \hat{a}_- \psi_n}$

If I had a normalized ψ_0 and I knew E_0, then I can build all $\psi_{n>0}$ and know E_n!!!

$$\psi_1 = \psi_{0+1} = \frac{1}{\sqrt{\frac{E_0}{\hbar\omega}+\frac{1}{2}}}\hat{a}_+\psi_0 \qquad E_1 = E_0+\hbar\omega$$

$$\psi_2 = \psi_{1+1} = \frac{1}{\sqrt{\frac{E_1}{\hbar\omega}+\frac{1}{2}}}\hat{a}_+\psi_1$$

$$= \frac{1}{\sqrt{\frac{E_1}{\hbar\omega}+\frac{1}{2}}}\hat{a}_+\frac{1}{\sqrt{\frac{E_0}{\hbar\omega}+\frac{1}{2}}}\hat{a}_+\psi_0$$

$\psi_0, E_0 = ?$

Recall from HW 2.2 if $E_n < V_{min}$ then ψ_n is BADD!!!

Assume $E_n > V_{min} \overset{SHO}{=} 0$

But $\hat{a}_- \psi_n \Rightarrow E_n - \hbar\omega = E_{n-1}$

This process has to bottom out!!

Impose $\underbrace{\hat{a}_- \psi_0 = 0}_{\text{Equation can be solved for } \psi_0} \Rightarrow \psi_{-1} = 0$

$\psi_{-2} = \hat{a}_- \psi_{-1} = 0$

$$\frac{1}{\sqrt{2m\hbar\omega}} (m\omega \hat{x} + i\hat{p})\psi_0 = 0$$

$$\frac{1}{\sqrt{2m\hbar\omega}} \underbrace{(m\omega x + \hbar \tfrac{d}{dx})\psi_0}_{=0} = 0 \Rightarrow \frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar} x \psi_0$$

$$\psi_0(x) = A e^{-\frac{m\omega}{2\hbar} x^2}$$

$$\int \psi_0^* \psi_0 \, dx = 1$$

$$\Downarrow$$

$$A = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}$$

Finding E_0: $\hat{H}\psi_0 = E_0 \psi_0$

$$\hbar\omega(\hat{a}_+ \underbrace{\hat{a}_-}_{0} + \tfrac{1}{2})\psi_0 = E_0 \psi_0$$

$$\tfrac{1}{2}\hbar\omega \psi_0 = E_0 \psi_0$$

$$E_0 = \tfrac{1}{2}\hbar\omega$$

Back to normalization of $\psi_{n+1} = \frac{1}{\sqrt{\frac{E_n}{\hbar\omega} + \frac{1}{2}}} \hat{a}_+ \psi_n$

$$\psi_1 = \frac{1}{\sqrt{\frac{E_0}{\hbar\omega} + \frac{1}{2}}} \hat{a}_+ \psi_0$$

$$\psi_1 = \hat{a}_+ \psi_0$$

$$\psi_2 = \frac{1}{\sqrt{\frac{E_1}{\hbar\omega} + \frac{1}{2}}} \hat{a}_+ \psi_1$$

$$= \frac{1}{\sqrt{2}} \hat{a}_+ \psi_1$$

$$\psi_n = \frac{1}{\sqrt{n}} \hat{a}_+ \psi_{n-1}$$

$$= \frac{1}{\sqrt{n!}} \hat{a}_+^n \psi_0$$

Why did this work?

$$\underbrace{-\frac{\hbar^2}{2m}\frac{\partial^2\psi_n}{\partial x^2}+\frac{1}{2}m\omega^2 x\psi_n = E_n\psi_n}_{\text{This sucks!!}}$$

We "factorized" $\hat{H} = \hbar\omega(\hat{a}_+\hat{a}_- + \frac{1}{2})$

$$\hat{H}\psi_0 = E_0\psi_0$$

$$\hbar\omega(\cancelto{0}{\hat{a}_+\hat{a}_-} + \tfrac{1}{2})\psi_0 = E_0\psi_0$$

$$\hat{a}_-\psi_0 = 0$$