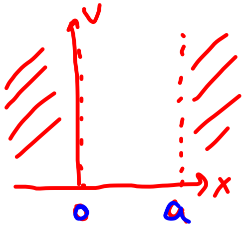
Review: TDSE $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + \underline{\underline{V(x)}}\,\Psi$ $\quad \Psi(x,t)$

big daddy psi
total w.f.

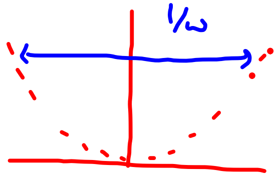
Assume: $\Psi(x,t) = \sum_n c_n \underbrace{\psi_n(x,t)}_{\phi_n(t)\,\psi_n(x)}$

$e^{-i\frac{E_n}{\hbar}t}$ $\quad$ TISE $\hat{H}\psi_n = E_n\psi_n$



$\psi_n(x) = \sqrt{\frac{2}{a}}\,\sin\left(\frac{n\pi}{a}x\right)$

$E_n = \frac{\pi^2\hbar^2}{2ma^2}\,n^2$



$\frac{1}{2}m\omega^2 x^2$

$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{n!}}(\hat{a}_+)^n e^{-\frac{m\omega}{2\hbar}x^2}$

$E_n = (n+\frac{1}{2})\hbar\omega$ $\quad \hat{a}_+ = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} - i\hat{p})$

What is QM? TDSE
TISE
$\psi(x,t)$
$[\hat{x},\hat{p}] = i\hbar$

The free particle $V(x) = 0$ for all x

TISE $\hat{H}\psi_k = E_k\psi_k$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_k}{dx^2} = E_k\psi_k$$

$$\frac{d^2\psi_k}{dx^2} = -\frac{2mE_k}{\hbar^2}\psi_k \Rightarrow \psi_k = \overbrace{Ae^{ikx} + Be^{-ikx}}^{Ae^{ikx}\quad k \text{ is } \pm}$$

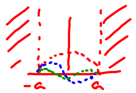
$$\underbrace{k^2 = \frac{2mE_k}{\hbar^2}}$$

$$E_k = \frac{\hbar^2 k^2}{2m}$$

It all comes down to k. What is k?

Spatial boundary conditions $\Rightarrow k$

Infinite well



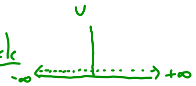
$\psi(x) = 0$ outside of well for $E < V_{min}$ normalizable ψ

Continuity: $\psi(-a) = \psi(a) = 0$

$k = \frac{n\pi}{a}$ discrete spectrum for k

Smoothness? No (∞ jumps in V eliminate smoothness)

Free particle



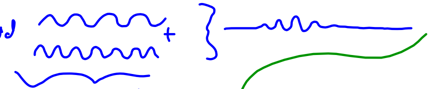
We need Ψ to be normalizable!

Try to normalize $\psi(x) \Rightarrow \int_{-\infty}^{\infty} \psi^*\psi \, dx = 1$

$\int_{-\infty}^{\infty} A^2 \, dx = 1$ Not-normalizable

But there is some magic you can do: Normalizable

You can add



non-normalizable

k must be continuous

Contrast : ∞-well   free particle

Back to basics: $\boxed{\begin{array}{l} \psi(k,x) \qquad k \in \mathbb{R} \ (\pm \text{ reals}) \\ E(k) = \dfrac{\hbar^2 k^2}{2m} \end{array}}$

k-continuous changes things:

Getting back to normalization: By "adding" up e^{ikx} w/ diff. ks you can get a normalizable wave-packet

crap!

$$\cancel{\psi(x) = \sum_k c_k \psi(k,x)} \qquad \cancel{\psi(x) = \int c(k) \psi(k,x)\, dk}$$

Discrete: $\Psi(x,t) = \sum_n c_n e^{-i\frac{E_n}{\hbar}t} \psi_n(x)$

Cont.: $\Psi(x,t) = \int_{-\infty}^{\infty} c(k) \underbrace{e^{-i\frac{E(k)}{\hbar}t}}_{\phi(k,t)} \psi(k,x)\, dk$

$$\downarrow$$

$\dfrac{\phi(k)}{\sqrt{2\pi}}$ (bad choice)

How do we find $\phi(k)$? Initial conditions

Discrete case: $\Psi(x,0)$ $\qquad \Psi(x,t) = \sum_n c_n \psi_n(x,t)$

$$c_n = \int_{-\infty}^{\infty} \psi_n^*(x,t)\, \Psi(x,t)\, dx$$

$$= \int_{-\infty}^{\infty} \psi_n^*(x)\, \phi_n^*(t) \left[\sum_m c_m \phi_m(t) \psi_m(x) \right] dx$$

$$= \sum_m c_m \phi_m^* \phi_m \int_{-\infty}^{\infty} \psi_n^*(x) \psi_m(x)\, dx$$

$$\delta_{nm} = \begin{cases} 1 & m=n \\ 0 & m \neq n \end{cases}$$

$$= c_n \cancel{\phi_n^* \phi_n}^{\,1}$$

$$= c_n$$

Continuous: $$\frac{\phi(k)}{\sqrt{2\pi}} = \int_{-\infty}^{\infty} \psi^*(k,x,t)\, \Psi(x,t)\, dx$$

$$= \int_{-\infty}^{\infty} \left[e^{-i(kx - \frac{\hbar k^2}{2m} t)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k') e^{i(k'x - \frac{\hbar k'^2}{2m} t)}\, dk' \right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k') e^{-i(-\frac{\hbar k^2}{2m} + \frac{\hbar k'^2}{2m})t} \underbrace{\left[\int_{-\infty}^{\infty} e^{i(k-k')x}\, dx \right]}_{\text{Dirac delta function}} dk'$$

$$\delta(k-k') = \begin{cases} \infty & k=k' \\ 0 & k \neq k' \end{cases}$$

$$\boxed{\int f(k)\, \delta(k-k')\, dk = f(k')}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k') e^{-i\frac{\hbar}{2m}(k'^2 - k^2)t}\, \delta(k-k')\, dk'$$

$$= \frac{\phi(k)}{\sqrt{2\pi}}$$