Exam 1 info: The "Starbucks" model

Short : 8 short questions, little to no calculation

Tall : Choose 2 of 4 intermediate calculations.

"Thank you"

GrAndE : Choose 1 of 2 looooong calculations.

Review: In QM we have

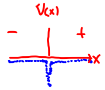
Normalizable Bound States (discrete spectrum) - for $E_{tot} < V(\pm\infty) \equiv 0$

Non-normalizable Scattering States (cont. spectrum) - for $E_{tot} > V(\pm\infty) \equiv 0$

Note: $E_{tot} > V_{min}$ for normalizable bound states

Focus on bound states of δ-well and finite-well.

Scattering states after exam.



<u>δ-well</u> $V(x) = -\alpha\,\delta(x)$

Assume $E<0$ for bound states.

$$\int_{-\infty}^{\infty} \alpha\,\delta(x)\,dx = 1\cdot\alpha$$

TISE: $\dfrac{d^2\psi_n^{\pm}}{dx^2} = k^2\psi_n^{\pm} \Rightarrow \psi_n^{\pm}(x) = A_{\pm}e^{kx} + B_{\pm}e^{-kx}$

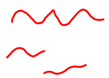
$k^2 = -\dfrac{2mE}{\hbar^2}$

For finite solution @ $x = \pm\infty$

$A_+ = B_- = 0$

Continuity @ $x=0$: $\psi_n^-(0) = \psi_n^+(0)$

$$B_+ = A_- = A \Rightarrow \underbrace{\psi_n(x) = Ae^{-k|x|}}_{2 \text{ unknowns } A,k}$$



Need 2 constraints:

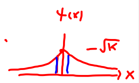
Normalization $\displaystyle\int_{-\infty}^{\infty}\psi^*\psi\,dx = 1 = 2A^2\int_0^{\infty}e^{-2kx}\,dx$

$$= -\frac{A^2}{k}e^{-2kx}\Big|_0^{\infty}$$

$$= \frac{A^2}{k} \Rightarrow A = \sqrt{k}$$

Need 1 more:

Can't demand $\dfrac{d\psi}{dx}$ be continuous but...



$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\left[-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2} - \alpha\,\delta(x)\psi_n = E_n\psi\right]dx$$

$$\lim_{\epsilon\to 0}\left[-\frac{\hbar^2}{2m}\frac{d\psi_n}{dx}\Big|_{-\epsilon}^{+\epsilon} - \alpha\,\psi_n(0) = E_n\cdot\text{Area under }\psi\Big|_{-\epsilon}^{\epsilon}\right]$$

$\sqrt{k}$ $\quad 0$

$$-\frac{\hbar^2}{2m}\left(\frac{d\psi_n^+}{dx}\Big|_{0^+} - \frac{d\psi_n^-}{dx}\Big|_{0^-}\right) - \alpha A = 0$$

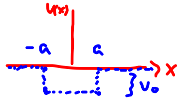
$$\left(\frac{d\psi_n^+}{dx} - \frac{d\psi_n^-}{dx}\right)\Big|_0 = -\frac{2m\alpha A}{\hbar^2}$$

$$-k\cancel{A}\cancel{e^0}^{\,1} - k\cancel{A}\cancel{e^0}^{\,1} = -\frac{2m\alpha}{\hbar^2}\cancel{A}$$

$$k = \frac{m\alpha}{\hbar^2} \qquad k^2 = -\frac{2mE}{\hbar^2}$$

$$\psi(x) = \sqrt{\frac{m\alpha}{\hbar^2}}\,e^{-\frac{m\alpha}{\hbar^2}|x|} \qquad E = -\frac{\hbar^2k^2}{2m} = -\frac{m\alpha^2}{2\hbar^2}$$

finite-well



$\Rightarrow \ \ 0, \ -V_0$

$$V(x) = \begin{cases} 0 & x<-a & - \\ -V_0 & -a<x<a & 0 \\ 0 & x>a & + \end{cases}$$

TISE:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n^\pm}{dx^2} = E_n\psi_n^\pm$$

$$\underbrace{-\frac{\hbar^2}{2m}\frac{d^2\psi_n^0}{dx^2} - V_0\psi_n^0 = E_n\psi_n^0}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n^0}{dx^2} = (E_n+V_0)\psi_n^0$$

Assume:
$E_n<0$ (bound states)
$\underbrace{E_n > -V_0 \text{ (normalizable)}}_{(E_n+V_0)>0}$

$-\frac{2mE_n}{\hbar^2} \equiv k^2$

$$\frac{d^2\psi_n^\pm}{dx^2} = k^2\psi_n^\pm$$

$$\frac{d^2\psi_n^0}{dx^2} = -\ell^2\psi_n^0$$

$\frac{2m(E_n+V_0)}{\hbar^2} \equiv \ell^2$

Again: $\psi_n^\pm(x) = A_\pm e^{kx} + B_\pm e^{-kx} \Rightarrow A_+ = B_- = 0$
for normalizability

Middle region: $\psi_n^0(x) = C\sin(\ell x) + D\cos(\ell x)$ [No loss of generality!]
ψ, ψ^* both satisfy

But... Notice when C and D are not $=0$, $\psi_n^0(x)$ is not even or odd.

If $V(x) = V(-x)$ you can use only E and O solutions w/out loss of generality.

All solutions are contained in: $\psi_n^0(x) = C\sin(\ell x)$
$\psi_n^0(x) = D\cos(\ell x)$

Book does even solutions, I'll do odd.

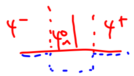
A fully odd solution will involve: $\psi_n^- = A_- e^{kx}$ — $A_- = -B_+$ for odd

already odd $\longrightarrow$ $\psi_n^o = C\sin(\ell x)$

$\psi_n^+ = B_+ e^{-kx}$

$\psi_n^- = A e^{kx}$

$\psi_n^+ = -A e^{-kx}$



No new info.

A, C, ℓ, k are unknowns

Continuity @ $x=a$ $\quad \psi_n^o(a) = \psi_n^+(a)$ $\qquad$ @ $x=-a$ $\quad \psi_n^o(-a) = \psi_n^-(-a)$

(i) $C\sin(\ell a) = -Ae^{-ka}$ $\qquad$ $C\sin(-\ell a) = Ae^{-ka}$

$-C\sin(\ell a) = Ae^{-ka}$

Smoothness @ $x=a$ $\quad \left.\frac{d\psi_n^o}{dx}\right|_a = \left.\frac{d\psi_n^+}{dx}\right|_a$

(ii) $\ell C\cos(\ell a) = kAe^{-ka}$

Divide $\frac{(ii)}{(i)} \Rightarrow$ $\ell\cot(\ell a) = -k$

Include $\quad k^2 = -\frac{2mE_n}{\hbar^2}$

$\ell^2 = \frac{2m(E_n+V_0)}{\hbar^2}$

$\Big\}$ 3 equations, 3 unknowns (ℓ, k, E_n)

Some trickery:

Consider: $k^2+\ell^2 = \frac{2mV_0}{\hbar^2} \Rightarrow k = \sqrt{\frac{2mV_0}{\hbar^2} - \ell^2}$

$ka = \sqrt{\frac{2mV_0a^2}{\hbar^2} - (\ell a)^2}$

$= \sqrt{z_0^2 - z^2}$ $\qquad z \equiv \ell a$

$z_0 = \sqrt{\frac{2mV_0a^2}{\hbar^2}}$

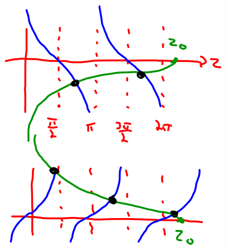
Now: $\cot(\ell a) = -\frac{k}{\ell} = -\frac{ka}{\ell a}$

$$\underbrace{\cot(z) = -\frac{\sqrt{z_0^2-z^2}}{z}}_{\text{Solve this!}} = -\sqrt{\left(\frac{z_0}{z}\right)^2 - 1}$$

Plot $\cot(z)$ w/ $-\sqrt{\left(\frac{z_0}{z}\right)^2-1}$:

Even solutions $\tan(z)$ w/ $\sqrt{\left(\frac{z_0}{z}\right)^2-1}$

$z_0 \equiv \sqrt{\frac{2mV_0a^2}{\hbar^2}}$



Jackin' with d' well:

(i) wide (a large) and deep (V_0 large) $\Rightarrow z_0$ large

Odd solutions $z = n\pi$ all n $\Rightarrow E_n + V_0 = \frac{n^2\pi^2\hbar^2}{a^2 2m} = \frac{(2n)^2\pi^2\hbar^2}{(2a)^2 2m}$ $\Big\}$ odd solutions to ∞-well of width $2a$

Even solutions $z = \frac{n\pi}{2}$ n-odd $\Rightarrow E_n + V_0 = \frac{n^2\pi^2\hbar^2}{(2a)^2 2m}$ $\Big\}$ even solutions to ∞-well of width $2a$

$z = \ell a$

(ii) narrow (a small) and shallow (V_0 small) $\Rightarrow z_0$ small $\Big\}$ As $z_0 \to 0$ only 1 solution and it's even!