Review: Normalizable Ψ's live in a Hilbert space

a) $L_2(a,b)$: $\int_a^b |f|^2 dx < \infty$

b) We have an inner product

$$\langle f|g\rangle = \int_a^b f^* g \, dx$$

Construct a linearly independent, orthonormal basis which spans $L_2(a,b)$.

Is orthogonality and linear ind. the same?

Orthogonality $\Rightarrow$ lin. ind. $\begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix}$

Linear Ind. $\not\Rightarrow$ orthogonality $\begin{pmatrix}1\\1\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix}$

$\langle f|$ often called the dual of $|f\rangle$.

$$\underbrace{\langle f}_{\text{include complex conjugation}} \,|\, \underbrace{f}_{\text{complex}} \rangle = 1$$

So far we have a vector space.
No we want to look at transformations.

CM: $\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} \Rightarrow R_z \vec{v} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{\text{generally don't commute}} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \vec{v}'$

We should expect transformations to come from operators.

We have seen: $\hat{x}, \hat{p}, \hat{H}, \hat{a}_\pm$

2 categories: a) "Formal" constructions, e.g. $\hat{a}_\pm$
b) "Physical" operators (observables)

<u>Important</u> Operators corresponding to observables are always Hermitian, i.e.

$$\langle f | \hat{Q} g \rangle = \langle \hat{Q} f | g \rangle$$

for any f, g in the Hilbert space.

Why do we need this? Things we measure must be real, i.e. $\langle \hat{Q} \rangle = \langle \hat{Q} \rangle^*$.

$$\langle \hat{Q} \rangle_f = \int_a^b f^* (\hat{Q} f) dx = \langle f | \hat{Q} f \rangle$$

$$\langle \hat{Q} \rangle_f^* = \int_a^b f (\hat{Q}^* f^*) dx = \int_a^b (\hat{Q}^* f^*) f \, dx = \langle \hat{Q} f | f \rangle$$

Reality $\Rightarrow \langle f | \hat{Q} f \rangle = \langle \hat{Q} f | f \rangle$

Therefore (due to your sweat and tears and blood)
$\hat{Q}$ must be hermitian.

<u>Example</u> $\hat{p} = -i\hbar \frac{\partial}{\partial x}$ should be Hermitian

Is $\langle f | \hat{p} f \rangle \stackrel{?}{=} \langle \hat{p} f | f \rangle$

$$\langle f | \hat{p} f \rangle = \int_a^b f^* \underbrace{\left(-i\hbar \frac{\partial f}{\partial x}\right)}_{-i\hbar \frac{\partial}{\partial x}(f^* f) + i\hbar \frac{\partial f^*}{\partial x} f} dx = \cancel{-i\hbar f^* f \Big|_a^b}^{\,0} + \int_a^b \left(i\hbar \frac{\partial}{\partial x} f^*\right) f \, dx$$

$$= \int_a^b (\hat{p}^* f^*) f \, dx$$

$$= \langle \hat{p} f | f \rangle$$

$\hat{Q} = i3 \neq$ not Hermitian

In general we need $\begin{cases} i & \text{if an order of derivatives} \\ \text{no } i & \text{if an even of derivatives} \end{cases}$

$$\hat{x} = x \quad \hat{p} = -i\hbar \frac{\partial}{\partial x} \quad \hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$$

In considering $\hat{H}$ we found: $\Psi = \sum_n c_n \underbrace{\psi_n}_{\text{stationary states}}$

a) ψ_n are determinant, $\hat{H}\psi_n = E_n \psi_n$ w/ 100% prob.

b) ψ_n are orthogonal

<u>Fact</u> For Hermitian operators (observables) we always get a natural orthogonal basis of determinant states.

<u>Determinant states</u>: Pick a $\hat{Q}$ such that $\langle \hat{Q}f|f\rangle = \langle f|\hat{Q}f\rangle$

$$\sigma_{\hat{Q}}^2 = \langle Q^2\rangle - \langle Q\rangle^2 = \langle(\hat{Q} - \langle\hat{Q}\rangle)^2\rangle = \langle(\hat{Q}-q)^2\rangle$$

from ch. 1

$$= \langle f|(\hat{Q}-q)^2 f\rangle$$

$$= \langle(\hat{Q}-q)f|(\hat{Q}-q)f\rangle$$

Let us define $\langle\hat{Q}\rangle = q$

If $\hat{Q}$ is Herm. $\Rightarrow$ $\hat{Q}+q$ is Herm.

For a determinant state we need $\sigma^2 = 0$

$$\langle(\hat{Q}-q)f|(\hat{Q}-q)f\rangle = 0$$


$\leftarrow \sigma^2 = 0$

$$\Downarrow$$

$$(\hat{Q}-q)f = 0$$

$$\underbrace{\hat{Q}f = qf}$$

e.g. $\hat{H}\psi = E\psi$

Bottom Line: Finding determinant states is the same as solving for eigenvectors of a given operator.

Different operators will generally generate set of determinant states.

∞-well $\quad \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right) \qquad \hat{H}_\infty \psi_n = E_n \psi_n$

$$\hat{p}\psi_n = -i\hbar\frac{\partial}{\partial x}\psi_n = -i\hbar\sqrt{\frac{2}{a}}\,\frac{n\pi}{a}\cos\left(\frac{n\pi}{a}x\right)$$

$$\neq (\ \ )\,\psi_n(x)$$

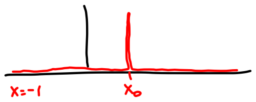
Determinant States of Position and Momentum

$\hat{p}$: $-i\hbar \frac{\partial}{\partial x} f_p(x) = p\, f_p(x)$

$$\frac{\partial f_p(x)}{\partial x} = \frac{ip}{\hbar} f_p(x) \Rightarrow \boxed{f_p(x) = A e^{\frac{ip}{\hbar}x}}$$

$\hat{x}$: $x\, g_{x_0}(x) = x_0\, g_{x_0}(x) \Rightarrow \boxed{g_{x_0}(x) = A\,\delta(x - x_0)}$

why?



But: These states do not live in $L_2(-\infty,\infty)$!!

$$\int_{-\infty}^{\infty} f_p^* f_p\, dx = |A|^2 \int_{-\infty}^{\infty} dx = \infty$$

$$\int_{-\infty}^{\infty} g_{x_0}^*(x)\, g_{x_0}(x)\, dx = |A|^2 \int_{-\infty}^{\infty} \underbrace{\delta(x_0 - x)}_{f(x)}\, \delta_0(x_0 - x)\, dx$$

$$= |A|^2 f(x_0)$$

$$= |A|^2 \delta(x_0 - x_0) = \infty$$

Buttt!! They are orthogonal.

$A = \frac{1}{\sqrt{2\pi\hbar}}$

$$\int_{-\infty}^{\infty} f_{p'}^* f_p\, dx = |A|^2 \int_{-\infty}^{\infty} e^{\frac{i(p-p')}{\hbar}x} dx = \overbrace{|A|^2 2\pi\hbar}\, \delta(p - p')$$

$= 0$ if $p \neq p'$

$$\int_{-\infty}^{\infty} g_{x_0'}^*(x)\, g_{x_0}(x)\, dx = |A|^2 \int_{-\infty}^{\infty} \underbrace{\delta(x_0' - x)}_{f(x)}\, \delta(x_0 - x)\, dx$$

$$= \underbrace{|A|^2}_{=1} \delta(x_0' - x_0) = 0 \text{ if } x_0' \neq x_0$$