Review: How to calculate probabilities associated with measurements for a given state $\Psi(x,t)$ in QM.

i) $\Psi(x,t)$ (governed by $\underbrace{\text{TDSE}}$) exists before we make measurement.
↳ defines the system through $V(x)$ and "content"

ii) Decide what "observable" to measure (energy, momentum, position, etc.)

iii) Form the corresponding hermitian ($\hat{H}, \hat{p}, \hat{x}$, etc.)

$\hat{Q} f_n(x) = q_n f_n(x)$

iv) Expand $\Psi(x,t)$ in the eigenstates (determinant states) of chosen operator (they orthonormal complete basis).

$\Psi = \sum_n c_n f_n(x)$

v) Expansion coefficients c_n give probabilities $|c_n|^2$ of any outcome.

vi) Expectation values follow: $\langle \hat{Q} \rangle = \sum_n |c_n|^2 q_n$

Extend Page

Note: We have seen $\langle \hat{H} \rangle$, $\langle \hat{p} \rangle$, $\langle \hat{x} \rangle$ but the only one we could express as $\langle \hat{H} \rangle = \sum_n |c_n|^2 E_n$

You can calculate these w/ eigenfunctions of the $\hat{H}$

v) Expect $\sum_n |c_n|^2 = 1$

Start off with a normalized Ψ: $\overbrace{\langle \Psi | \Psi \rangle}^{\int \Psi^* \Psi dx} = 1$

$$= \langle (\sum_n c_n f_n) | (\sum_{n'} c_{n'} f_{n'}) \rangle$$

$$= \sum_n \sum_{n'} c_n^* c_{n'} \underbrace{\langle f_n | f_{n'} \rangle}_{\delta_{nn'}}$$

$$= \sum_n |c_n|^2$$

vi) $\langle \hat{Q} \rangle = \langle \Psi | \hat{Q} \Psi \rangle = \langle \hat{Q} \Psi | \Psi \rangle$ use f_n's from $\hat{Q} f_n = q_n f_n$

$$= \langle (\sum_n c_n f_n) | \hat{Q} (\sum_{n'} c_{n'} f_{n'}) \rangle$$

$$= \sum_n \sum_{n'} c_n^* c_{n'} q_{n'} \underbrace{\langle f_n | f_{n'} \rangle}_{\delta_{nn'}}$$

$$= \sum_n |c_n|^2 q_n$$

Note: Both results rely on orthonormality of f_n.

The Uncertainty Principle

Common forms
$$\begin{cases} \Delta x \Delta p \geq \frac{\hbar}{2} \Rightarrow \sigma_x \sigma_p \geq \frac{\hbar}{2} & \text{(today we derive it)} \\ \Delta E \Delta t \geq \frac{\hbar}{2} \Rightarrow & \text{(today we derive it and understand)} \end{cases}$$

The $\overbrace{\text{"generalized uncertainty principle"}}^{\text{GUP}}$ can be stated for any 2 hermitian operators (observables).

2 things: a) It is sometimes trivial, e.g. $\sigma_x \sigma_y \geq 0$

b) It does <u>not</u> directly include Δt (there is no σ_t).

The GUP relies on:

i) Schwarz inequality: $\overbrace{\left| \int_a^b f^* g \, dx \right|}^{\vec{f}\cdot\vec{g}} \leq \overbrace{\sqrt{\int_a^b f^* f \, dx \int_a^b g^* g \, dx}}^{|\vec{f}||\vec{g}|}$

complex or $|\langle f | g \rangle| \leq \sqrt{\langle f | f \rangle \langle g | g \rangle}$

ii) $\boxed{|z|^2} = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2 \geq [\mathrm{Im}(z)]^2 = \boxed{\left[\frac{1}{2i}(\underbrace{z - z^*}_{2iv})\right]^2}$

$u + iv$
$u - iv$

iii) Hermiticity

Pick 2 observables $\hat{A}, \hat{B}$ and form: $f = (\underbrace{\hat{A} - \langle \hat{A} \rangle}_{\text{this is hermitian}}) \Psi$

$g = (\hat{B} - \langle \hat{B} \rangle) \Psi$

hermitian

this is the system

Then $\sigma_A^2 = \langle f | f \rangle$, $\sigma_B^2 = \langle g | g \rangle$

Using: i) $\sigma_A^2 \sigma_B^2 = \langle f | f \rangle \langle g | g \rangle \geq |\langle f | g \rangle|^2$

ii) $|\langle f | g \rangle|^2 \geq \left[\frac{1}{2i}(\langle f | g \rangle - \langle g | f \rangle)\right]^2$

Then: $\sigma_A^2 \sigma_B^2 \geq \left[\frac{1}{2i}(\langle f | g \rangle - \langle g | f \rangle)\right]^2$

can show: $\left.\begin{array}{l} \langle f | g \rangle = \langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle \\ \langle g | f \rangle = \langle \hat{B}\hat{A} \rangle - \langle \hat{B} \rangle \langle \hat{A} \rangle \end{array}\right\}$ look in book!

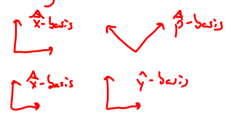
$$\langle f | g \rangle - \langle g | f \rangle = \langle \hat{A}\hat{B} \rangle - \cancel{\langle \hat{A} \rangle \langle \hat{B} \rangle} - \langle \hat{B}\hat{A} \rangle + \cancel{\langle \hat{B} \rangle \langle \hat{A} \rangle}$$
$$= \langle [\hat{A}, \hat{B}] \rangle$$

Finally: $\sigma_A^2 \sigma_B^2 \geq \left[\frac{1}{2i}\langle [\hat{A}, \hat{B}] \rangle\right]^2$ The G.U.P.

Examples: $[\hat{x}, \hat{p}] = i\hbar \Rightarrow \sigma_x^2 \sigma_p^2 \geq \frac{\hbar^2}{4} \Rightarrow \sigma_x \sigma_p \geq \frac{\hbar}{2}$

$[\hat{x}, \hat{y}] = 0 \Rightarrow \sigma_x \sigma_y \geq 0$

So uncertainty arises from noncommutativity of hermitian operators. In linear algebra, non-commutativity of operators corresponds to them <u>not</u> having a simultaneous basis of eigenvectors.



Now that we have GUP let's get ETUP!!

The ETUP relies on:

i) The GUP for $\hat{H}$ w/ some other $\hat{Q}$

ii) The TDSE which governs how Ψ (hence $\langle \hat{Q} \rangle_\Psi$) evolves w/ time.

Δt

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi \Rightarrow \frac{\partial \Psi}{\partial t} = \frac{1}{i\hbar}\hat{H}\Psi$$

iii) Hermiticity

First: $\frac{d}{dt}\langle \hat{Q} \rangle = \frac{d}{dt}\langle \Psi | \hat{Q}\Psi \rangle$

$$= \left\langle \frac{\partial \Psi}{\partial t} \middle| \hat{Q}\Psi \right\rangle + \left\langle \Psi \middle| \hat{Q}\frac{\partial \Psi}{\partial t} \right\rangle + \left\langle \Psi \middle| \frac{\partial \hat{Q}}{\partial t}\Psi \right\rangle$$

Use TDSE: $= -\frac{1}{i\hbar}\langle \hat{H}\Psi | \hat{Q}\Psi \rangle + \frac{1}{i\hbar}\langle \Psi | \hat{Q}\hat{H}\Psi \rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle$

Use hermiticity: $\langle \hat{H}\Psi | \hat{Q}\Psi \rangle = \langle \Psi | \hat{H}(\hat{Q}\Psi) \rangle$

Then: $\boxed{\frac{d\langle \hat{Q} \rangle}{dt} = \frac{i}{\hbar}\langle [\hat{H}, \hat{Q}] \rangle} + \cancel{\left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle}$ usually $= 0$

Now we GUP w/ $\hat{A} = \hat{H}$, $\hat{B} = \hat{Q}$: $\sigma_H^2 \sigma_Q^2 \geq \left[\frac{1}{2i}\langle [\hat{H}, \hat{Q}] \rangle\right]^2$

$$\sigma_H^2 \sigma_Q^2 \geq \left[\frac{\hbar}{2}\frac{d\langle \hat{Q} \rangle}{dt}\right]^2$$

Then: $\boxed{\sigma_H \sigma_Q \geq \frac{\hbar}{2}\left|\frac{d\langle \hat{Q} \rangle}{dt}\right|}$

Define: $\underbrace{\sigma_{\hat{H}} \equiv \Delta E \quad \text{and} \quad \Delta t \equiv \frac{\sigma_Q}{\left|\frac{d\langle \hat{Q} \rangle}{dt}\right|}}_{\Delta E \Delta t \geq \frac{\hbar}{2}}$

For stationary states: $\frac{d\langle \hat{Q} \rangle}{dt} = 0 \Rightarrow \Delta t \to \infty$

$$\Delta E \geq \frac{\hbar}{2\infty} = 0$$

$$\Delta E = 0 = \sigma_E$$