Review:

G.U.P. $$\sigma_{\hat A}^2 \sigma_{\hat B}^2 \geq \Big[\overbrace{\frac{1}{2i}\langle[\hat A, \hat B]\rangle}^{\text{from Schwarz inequality and complex magnitude inequality}}\Big]^2$$ for hermitian $\hat A$ and $\hat B$

E.T.U.P. $$\overbrace{\underbrace{\sigma_H}_{\Delta E}\, \sigma_{\hat Q} \geq \frac{\hbar}{2}\left|\frac{d\langle \hat Q\rangle}{dt}\right|}^{\text{from G.U.P. + T.D.S.E.}}$$ for hermitian $\hat Q$

ΔE and $\Delta t = \dfrac{\sigma_{\hat Q}}{\left|\frac{d\langle \hat Q\rangle}{dt}\right|}$ $\Rightarrow$ $\Delta E \Delta t \geq \dfrac{\hbar}{2}$

Minimal Uncertainty

We want Schwarz and complex magnitude equalities.

$$\langle f|f\rangle\langle g|g\rangle = |\langle f|g\rangle|^2 \Rightarrow g(x) = \underbrace{c}_{\text{complex constant}} f(x)$$

$$|\vec f||\vec g| \geq \vec f\cdot\vec g$$

$$fg \geq fg\underbrace{\cos\theta}_{}$$

for equality $\cos\theta = 1 \Rightarrow 0$

$$|g\rangle = c|f\rangle \qquad \langle g| = c^*\langle f|$$

$$\langle f|f\rangle c^*c\langle f|f\rangle = |c\langle f|f\rangle|^2 = |c|^2\langle f|f\rangle^2$$

Complex magnitude equality: $|z|^2 = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2 \geq [\mathrm{Im}(z)]^2$

$$|z|^2 = [\mathrm{Im}(z)]^2$$

$$z = \langle f|g\rangle$$

$$(\mathrm{Re}\langle f|g\rangle) = 0$$

$$(\mathrm{Re}\; c\,\underbrace{\langle f|f\rangle}_{\text{real}})$$

must be pure imaginary

$c = ia$

$$g(x) = ia f(x)$$

Recall: $\underbrace{\sigma_{\hat A}^2 = \langle f|f\rangle \quad \sigma_{\hat B}^2 = \langle g|g\rangle \quad \langle[\vec A, \vec B]\rangle = \langle f|g\rangle - \langle g|f\rangle}_{\text{all of this came from}}$

$$\boxed{\begin{aligned} f &= (\hat A - \langle \hat A\rangle)\Psi \\ g &= (\hat B - \langle \hat B\rangle)\Psi \end{aligned}}$$

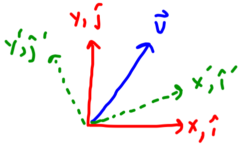
$$\Downarrow$$

$$(\hat B - \langle \hat B\rangle)\Psi = ia(\hat A - \langle \hat A\rangle)\Psi$$

Example: $\hat A = \hat x$, $\hat B = \hat p$

$$\underbrace{(\hat p - \langle \hat p\rangle)\Psi = ia(\hat x - \langle \hat x\rangle)\Psi}_{\text{differential equation for minimal uncertainty state } \Psi_{x,p}}$$

Dirac Notation



$$\vec{v} = v_x \hat{i} + v_y \hat{j} = v_x' \hat{i}' + v_y' \hat{j}'$$

Note: $v_x = \hat{i}\cdot\vec{v}$

$v_x' = \hat{i}'\cdot\vec{v}$

$$\vec{v} = (\hat{i}\cdot\vec{v})\hat{i} + (\hat{j}\cdot\vec{v})\hat{j} = (\hat{i}'\cdot\vec{v})\hat{i}' + (\hat{j}'\cdot\vec{v})\hat{j}'$$

We know each hermitian in QM generates an orthonormal basis.

In QM our state is a vector in a Hilbert space $\underbrace{|\Psi(t)\rangle}$

(kets are vector)

What is a bra? We know $(\underbrace{\vec{v}\cdot}_{\text{"dual" vector}})\vec{w} = \#$ (vector)

$$(\quad)(\,|\,) = \#$$

$$(\langle f|)|g\rangle = \langle f|g\rangle = \#$$

Now pick a $\hat{Q}$ and label its basis: $\hat{x}: |x\rangle$ (eigenvectors) (labelled by eigenvalue)

$\hat{p}: |p\rangle$

$\hat{H}: |E_n\rangle = |n\rangle$

These are like $\hat{i}, \hat{j}$

$$\langle p|p'\rangle = \delta(p-p')$$

$$\langle n|n'\rangle = \delta_{nn'}$$

Then: $v_x = (\hat{i}\cdot)\vec{v} \Rightarrow \langle x|\Psi(t)\rangle = \Psi(x,t)$

$$\langle p|\Psi(t)\rangle = \phi(p,t)$$

$$\langle n|\Psi(t)\rangle = c_n(t)$$

$$|\Psi(t)\rangle = \int \Psi(x,t)\overbrace{\delta(x-y)}^{|y\rangle}dy = \int \phi(p,t)\overbrace{\frac{1}{\sqrt{2\pi\hbar}}e^{\frac{ipx}{\hbar}}}^{|p\rangle}dp$$

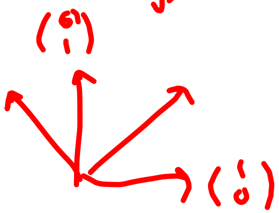
$$= \sum \underbrace{c_n e^{-i\frac{E_n}{\hbar}t}}_{c_n(t)} \underbrace{\psi_n(x)}_{|n\rangle}$$

Two state discussion in book:

$\hat{H} = \begin{pmatrix} h & g \\ g & h \end{pmatrix}$ $\quad$ $\hat{H}^\dagger = \hat{H}$ $\quad$ $\langle \hat{H} f | g \rangle = \langle f | \hat{H} g \rangle$

$\dagger$ — complex conjugate transpose

g, h are real



$\langle f | \hat{H} | g \rangle$

$\langle f | (\hat{H} | g \rangle)$

$(\langle f | \hat{H}) | g \rangle$

$\underbrace{(\langle f | \hat{H})}_{(\hat{H} | f \rangle)^\dagger}$

$\Downarrow$

finds eigenvectors of $\hat{H}$

$$|a_\pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow |\psi(t)\rangle$$

$|\psi(0)\rangle$

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle = \cancel{E |\psi(t)\rangle}$$

More on operators

Consider: $(\vec{v}\cdot)\vec{w} = \#$ order matters $\vec{w}\vec{v}\cdot = ?$

$$(\vec{w}\vec{v}\cdot)\vec{u} = \vec{w}(\overbrace{\vec{v}\cdot\vec{u}}^{\#}) = \#\vec{w} = \vec{u}'$$

$$\underbrace{(\vec{w}\vec{v}\cdot)}_{\text{operator}}\vec{u} = \vec{u}'$$

$$\langle f|g\rangle \qquad |g\rangle\langle f|\,|\alpha\rangle = \overbrace{\langle f|\alpha\rangle}^{\#}\,|g\rangle$$

Imagine: $\underbrace{\left(\hat{i}\hat{i}\cdot + \hat{j}\hat{j}\cdot + \hat{k}\hat{k}\cdot\right)}\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$

$$\mathbb{I} = \underbrace{\sum_B |B\rangle\langle B|}_{\text{resolution}}$$