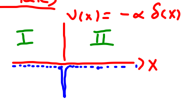
Scattering States

δ-well



Bound: $E < V(\pm\infty) = 0$ we found 1 solution $\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-\frac{m\alpha|x|}{\hbar^2}}$

$$E = -\frac{m\alpha^2}{2\hbar^2}$$

Scattering: $E > 0$

TISE: $\frac{d^2\psi}{dx^2} = \underbrace{-\frac{2mE}{\hbar^2}}_{-k^2}\psi \Rightarrow \psi_I(x) = Ae^{ikx} + Be^{-ikx}$

$$\psi_{II}(x) = \underbrace{Fe^{ikx} + Ge^{-ikx}}$$

could we sin, cos

but combining these w/

$\phi(k,t) = e^{-i\frac{E(k)}{\hbar}t}$ is easy.

Usually we need $\psi(\pm\infty) \neq \infty$. But not this time!

5 unknowns $A, B, F, G, k(E)$

Constraints: Continuity $\Rightarrow \psi^I(0) = \psi^{II}(0) \Rightarrow A + B = F + G$

"Smoothness" $\Rightarrow \left.\frac{d\psi^{II}}{dx}\right|_0 - \left.\frac{d\psi^I}{dx}\right|_0 = -\frac{2m\alpha}{\hbar^2}\psi(0)$

~~$x = \pm\infty$ behavior~~

~~Overall normalization~~

$$ik(F-G) - ik(A-B) = -\frac{2m\alpha}{\hbar^2}(A+B)$$

2 equations, 5 unknowns ... problem!!

Deep Thoughts

What do we expect for k? $\frac{d^2\psi}{dx^2} = \pm k^2\psi$

δ-well k unique

Typical V(x) (bound) k is discrete, e.g. $k = \frac{n\pi}{a}$

Typical V(x) (scattering) k is continuous

K is an input

a) What about A, B, F, G? we going to find a physical interpretation for these and then reformulate the problem w/ one of them = 0.

b) That leaves 3 unknowns, 2 equations: Change the question we ~~question~~ ask. Look for ratios.

a) Recall: $e^{i(\pm kx-\omega t)} = \cos(\underbrace{\pm kx-\omega t}_{}) + i\sin(\pm kx-\omega t)$

from $\psi(k,t) = e^{-i\frac{E(k)}{\hbar}t}$

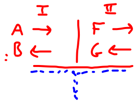
ϕ_0 - point of constant phase

$\Downarrow$ $\longrightarrow x$

$\phi_0 = \pm kx - \omega t \Rightarrow x = \pm\frac{\phi_0}{k} \pm \frac{\omega}{k}t$

for $\begin{cases} + & x \text{ grows w/ } t \text{ } \rightarrow \\ - & x \text{ dec. w/ } t \text{ } \leftarrow \end{cases}$

So we have:



General solution has all $\neq 0$.
But scattering experiments only involve incident, reflected, transmitted.

		incident	reflected	transmitted
Scattering from	left	A	B	F
	right	G	F	B

Note: A & G never appear together.

We will assume approach from left. $G=0$

b) 3 unknowns : Ask For an incident (I) wave, how much is transmitted (T) and how much is reflected? (R)

We set I : How to define R, T

$R = \frac{|B|^2}{|A|^2}$ $T = \frac{|F|^2}{|A|^2}$

Expect : $R+T=1$

How to find R, T :

C : $A+B = F + \cancel{G}^{\,0} \Rightarrow A+B=F$

"S" : $ik(F-\cancel{G}^{\,0}) - ik(A-B) = -\frac{2m\alpha}{\hbar^2}(A+B) \Rightarrow F-A+B = \frac{2m\alpha i}{k\hbar^2}(A+B)$

$F = A(1+2i\beta) - B(1-2i\beta)$

$\beta = \frac{m\alpha}{k\hbar^2}$

Combine to form: $B = \frac{i\beta}{1-i\beta}A$ $F = \frac{1}{1-i\beta}A$

$R = \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2}$ $T = \frac{|F|^2}{|A|^2} = \frac{1}{1+\beta^2}$

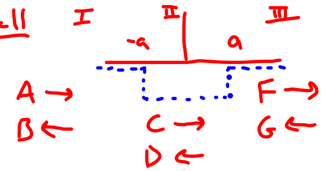
$R+T = \frac{\beta^2+1}{1+\beta^2} = 1$

Can express: $\beta(E) = \frac{m\alpha}{\sqrt{2mE}\,\hbar}$ $\beta^2 = \frac{m\alpha^2}{2E\hbar^2}$

$R = \frac{\frac{m\alpha^2}{2E\hbar^2}}{1+\frac{m\alpha^2}{2E\hbar^2}} = \frac{1}{\frac{2E\hbar^2}{m\alpha^2}+1}$ $T = \frac{1}{1+\frac{m\alpha^2}{2E\hbar^2}}$

Fix α, take $E\to\infty \Rightarrow R\to 0,\ T\to 1$
Fix E, take $\alpha\to\infty \Rightarrow R\to 1,\ T\to 0$

Finite-well



Assume $E > 0$

6 unknowns + $\cancel{k(E)}$

Consider scattering from left: $\underbrace{A, C, D, B, F}_{S}$

4 constraints: $C(-a)$, $C(a)$

$S(-a)$, $S(a)$

$$T = \frac{|F|^2}{|A|^2} = \left[1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)\right]^{-1}$$

$$R = \frac{|D|^2}{|A|^2}$$

As $E \to \infty$ fixed $V_0 \Rightarrow T \to 1$

As $V_0 \to \infty$ fixed $E \Rightarrow T \to 0$

When $\frac{2a}{\hbar}\sqrt{2m(E+V_0)} = n\pi \Rightarrow T = 1$