Review: Looking for two Hermitian operators whose eigenvalues break degeneracy of hydrogen energy levels $(\hat{H}, ?, ?)$

$n \quad \ell \quad m$

Found: $[\hat{H}, \hat{L}_z] = 0$ (even though $[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k$)

$[\hat{H}, \hat{L}^2] = 0$

$[\hat{L}^2, \hat{L}_z] = 0$

Maximally commuting set: $\hat{H}, \hat{L}^2, \hat{L}_z$

SHO: $[\hat{H}, \hat{p}] \neq 0$, $[\hat{H}, \hat{x}] \neq 0$ $\Big\}$ $\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} \mp i\hat{p})$

raise / lower eigenvalues of $\hat{H}$ $\quad \hat{a}_\pm \psi_n \to \psi_{n\pm 1}$

Ang. Mom.: $[\hat{L}_z, \hat{L}_x] \neq 0$, $[\hat{L}_z, \hat{L}_y] \neq 0$ $\Big\}$ $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$

raise / lower eigenvalues of $\hat{L}_z$

For SHO we set a lower bound $\hat{a}_-\psi_0 = 0$

For ang. mom. we will set an upper and lower bound.

We will need: $\left.\begin{array}{l}[\hat{L}_z, \hat{L}_\pm] = \pm\hbar \hat{L}_\pm \\ [\hat{L}^2, \hat{L}_\pm] = 0\end{array}\right\} \Leftarrow$ from $[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k$

Show that $\hat{L}_\pm$ change eigenvalue of $\hat{L}_z$:

Suppose $\hat{L}_z f = mf$, then consider $\hat{L}_\pm f$

$$\hat{L}_z(\hat{L}_\pm f) = L_\pm(L_z f) \pm \hbar L_\pm f = L_\pm(mf) \pm \hbar L_\pm f$$

$$L_z L_\pm - L_\pm L_z = \pm\hbar L_\pm \qquad = (m \pm \hbar) L_\pm f$$

So $\hat{L}_z f = mf$

$$\hat{L}_z(\hat{L}_\pm f) = (m \pm \hbar)\hat{L}_\pm f$$

Consider eigenvalues of $\hat{L}^2$:

Start with $\hat{L}^2 f = \lambda f$, then construct $\hat{L}_\pm f$

$$\hat{L}^2(\hat{L}_\pm f) = \hat{L}_\pm(\hat{L}^2 f) = \hat{L}_\pm \lambda f = \lambda(\hat{L}_\pm f)$$

Imposing upper and lower bounds

Expect $L_{z,max} \leq \sqrt{L^2}$ so:

$$\left.\begin{array}{l}\hat{L}_+ f_{top} = 0 \\ \hat{L}_- f_{bot} = 0\end{array}\right\} \text{let} \left\{\begin{array}{l}\hat{L}_z f_{top} = \hbar\ell f_{top} \\ \hat{L}_z f_{bot} = \hbar\bar{\ell} f_{bot}\end{array}\right.$$

You might expect $\hat{L}^2 f_{top} = \hbar^2\ell^2 f_{top} \Longleftarrow$ Wrong!!

Not same

Consider: $\hat{L}_\pm \hat{L}_\mp = (\hat{L}_x \pm i\hat{L}_y)(\hat{L}_x \mp i\hat{L}_y)$

$$= \hat{L}_x^2 + \hat{L}_y^2 \mp i\underbrace{(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x)}_{i\hbar\hat{L}_z}$$

$$= \hat{L}^2 - \hat{L}_z^2 \pm \hbar\hat{L}_z$$

$$\hat{L}^2 = \hat{L}_\pm\hat{L}_\mp + \hat{L}_z^2 \mp \hbar\hat{L}_z$$

Then:

$$\hat{L}^2 f_{top} = (\hat{L}_-\hat{L}_+ + \hat{L}_z^2 + \hbar\hat{L}_z) f_{top} = (0 + \hbar^2\ell^2 + \hbar^2\ell) f_{top}$$

$$= \hbar^2\ell(\ell+1) f_{top}$$

Eigenvalue of $\hat{L}^2$ is larger than max. eigenvalue of $\hat{L}_z$.

$$\hat{L}^2 f_{bot} = (\hat{L}_+\hat{L}_- + \hat{L}_z^2 - \hbar\hat{L}_z) f_{bot} = \hbar^2\bar{\ell}(\bar{\ell}-1) f_{bot}$$

But $\cancel{\hbar^2}\ell(\ell+1) = \cancel{\hbar^2}\bar{\ell}(\bar{\ell}-1) \Rightarrow \bar{\ell} = \ell+1$ (stoopid!)

$\Rightarrow \bar{\ell} = -\ell$

But $\hat{L}_\pm$ doesn't change the eigenvalue of $\hat{L}_z$ for any state.

So for any state on the ladder we have $\hat{L}^2 f = \hbar\,\ell(\ell+1) f$

ooo...aahh!!

Recall: $\hat{L}_z(\hat{L}_- f) = (m-\hbar)(\hat{L}_- f)$ where $\hat{L}_z f = mf$

In particular: $\hat{L}_z(\hat{L}_- f_{top}) = (\hbar\ell - \hbar)(\hat{L}_- f_{top}) = \hbar(\ell-1)(\hat{L}_- f_{top})$

Now repeat $\hat{L}_- \hat{L}_- f_{top}$, etc. etc.

$$
\begin{aligned}
f_{top} &\Rightarrow \hbar\ell \\
\hat{L}_- f_{top} &\Rightarrow \hbar(\ell-1) \\
\hat{L}_-^2 f_{top} &\Rightarrow \hbar(\ell-2) \\
&\vdots \qquad\quad \vdots \\
f_{bot} &\quad\;\; \hbar(\underbrace{\ell-N}) \\
\hat{L}_z f_{bot} &= -\hbar\ell
\end{aligned}
\Bigg\}
\quad
\begin{aligned}
&\hbar(\ell-N) = -\hbar\ell \\
&\ell = \frac{N}{2} \qquad N = 0, 1, 2, \cdots
\end{aligned}
$$

So what do have? f are eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$ w/

$$\hat{L}^2 f_\ell^m = \hbar^2 \ell(\ell+1) f_\ell^m \qquad \ell = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \cdots$$

$$\hat{L}_z f_\ell^m = \hbar m f_\ell^m \qquad m = -\ell, \cdots, +\ell$$

Look's a lot like $Y_\ell^m(\theta,\phi)$ except for $\ell = \frac{1}{2}$-integer

We defined the spherical harmonics as solutions to the angular part of TISE for $\hat{H}_{hyd}$.