Review: $\hat{S}^2 \chi_s^m = \hbar^2 s(s+1) \chi_s^m \qquad s = 0, \frac{1}{2}, 1, \cdots$

$\hat{S}_z \chi_s^m = m\hbar \chi_s^m \qquad m = -s, \cdots, s$

Example: spin-$\frac{1}{2}$ $\quad s = \frac{1}{2} \quad m = \pm\frac{1}{2}$

$\chi_{\frac{1}{2}}^{-\frac{1}{2}} = |\frac{1}{2}, -\frac{1}{2}\rangle = \chi_- = \downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\chi_{\frac{1}{2}}^{\frac{1}{2}} = |\frac{1}{2}, \frac{1}{2}\rangle = \chi_+ = \uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

eigenspinors of $\hat{S}_z$

$\hat{\vec{S}} = \frac{\hbar}{2} \hat{\vec{\sigma}} \qquad \hat{\vec{\sigma}} = \underbrace{\underset{\hat{\sigma}_x}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}, \underset{\hat{\sigma}_y}{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}, \underset{\hat{\sigma}_z}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}}_{\text{all hermitian}}$

$\hat{S}_\pm = \hat{S}_x \pm i \hat{S}_y$

$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$

$\chi_\pm^x = \begin{pmatrix} 1/\sqrt{2} \\ \pm 1/\sqrt{2} \end{pmatrix}$

Detecting Spin

A spinning charged object  generates a magnetic dipole $\vec{\mu} = \gamma \vec{S}$.

γ: gyromagnetic ratio

a. To detect spin dipole we Stern-Gerlach.

b. To understand SG we need Larmor precession.

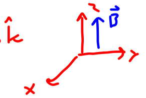
Larmor Precession

Magnetic dipole in $\vec{B}$-field feels a torque.

Fixing the position (no Lorentz force!).

$H = -\vec{\mu}\cdot\vec{B} = -\gamma \vec{S}\cdot\vec{B}$

Turn on $\vec{B} = B_0 \hat{k}$



$\hat{H} = -\gamma \hat{S}_z B_0$

$= -\frac{\gamma B_0 \hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

Then: $\hat{H}\chi_\pm = E_\pm \chi_\pm$ (TISE)

$E_\pm = \mp \frac{\gamma B_0 \hbar}{2}$ $\qquad \uparrow \quad \downarrow$

$E_+ < E_-$

$|a|^2 + |b|^2 = 1$

We know that if $\chi(t=0) = a\chi_+ + b\chi_-$

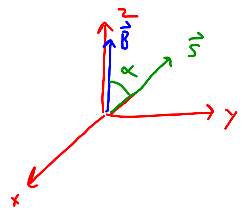
then $\chi(t) = a\chi_+ e^{-i\frac{E_+}{\hbar}t} + b\chi_- e^{-i\frac{E_-}{\hbar}t}$

$\chi_+ = \begin{pmatrix}1\\0\end{pmatrix}$, $\chi_- = \begin{pmatrix}0\\1\end{pmatrix}$

$= \begin{pmatrix} a e^{-i\frac{E_+}{\hbar}t} \\ b e^{-i\frac{E_-}{\hbar}t}\end{pmatrix} = \begin{pmatrix}\cos(\frac{\alpha}{2}) e^{-i\frac{E_+}{\hbar}t} \\ \sin(\frac{\alpha}{2}) e^{-i\frac{E_-}{\hbar}t}\end{pmatrix}$

$\langle \hat{S}_x \rangle_{\chi(t)} = \chi^\dagger(t)\hat{S}_x\chi(t) = \frac{\hbar}{2}\sin\alpha\cos(\gamma B_0 t)$

$\langle \hat{S}_y \rangle_{\chi(t)} = -\frac{\hbar}{2}\sin\alpha\sin(\gamma B_0 t)$ $\qquad \langle \hat{S}_z \rangle_{\chi(t)} = \frac{\hbar}{2}\cos\alpha$



$\omega_L = \gamma B_0$

γ: gyromagnetic ratio

B_0: magnetic field strength

Stern-Gerlach

$$\vec{F} = \vec{\nabla}(\vec{\mu}\cdot\vec{B}) = \vec{\nabla}(\gamma\vec{S}\cdot\vec{B})$$

constant contribution along z

Consider: $\vec{B} = -\alpha x\hat{i} + (B_0 + \alpha z)\hat{k}$ $\quad (\vec{\nabla}\cdot\vec{B} = 0)$

nonhomogeneous strength

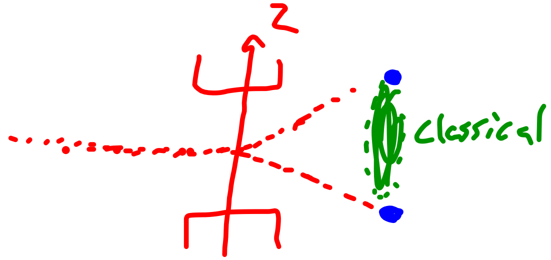
$$\vec{F} = \gamma\vec{\nabla}\left(-\alpha S_x x + S_z(B_0 + \alpha z)\right)$$

$$= \gamma\alpha(-S_x\hat{i} + S_z\hat{k})$$

$$= \gamma\alpha(-\cancel{\langle\hat{S}_x\rangle}\hat{i} + \langle\hat{S}_z\rangle\hat{k})$$

averages to zero for B_0 large due to Larmor precession.

$$\simeq \gamma\alpha\langle\hat{S}_z\rangle\hat{k}$$



Consider: 50% ↑ 50% ↓

$\vec{F} = 0$

100% ↑

$\vec{F} = \gamma\alpha\frac{\hbar}{2}\hat{k}$

100% ↓

$\vec{F} = -\gamma\alpha\frac{\hbar}{2}\hat{k}$