Review: $\boxed{i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}_{hyd} \Psi}$ $\hat{H}_{hyd} = \frac{\hat{p}^2}{2m} - \frac{e^2}{4\pi\epsilon_0} \frac{1}{r}$

$$\Psi(r,\theta,\phi,t) = \sum_n c_n e^{-i\frac{E_n}{\hbar}t} \underbrace{R_{n\ell}(r)\, Y_\ell^m(\theta,\phi)}_{\psi_{n\ell m}(r,\theta,\phi)}$$

$\psi_{n\ell m}(r,\theta,\phi)$ solves $\hat{H}_{hyd}\psi_{n\ell m} = E_n \psi_{n\ell m}$ (TISE)

generate an orthonormal basis in Hilbert space of bound states

ψ_{200} → $\psi_{n=2\ \ell=0\ m=0}$; $\psi_{n=200}$

$$\int \psi^*_{n\ell m} \psi_{n'\ell'm'}\, dV = \delta_{nn'}\delta_{\ell\ell'}\delta_{mm'}$$

To solve we first separated $R(r)$ from Θ, Φ using $\ell(\ell+1)$.
($R(r) \to R_{n\ell}$, $\Theta,\Phi \to Y_\ell^m$)

Y_ℓ^m - known functions (spherical harmonics) - you give me a value of ℓ, m I give $Y_\ell^m(\theta,\phi)$.

$R_{n\ell}(r)$ - learned how to build

$$R_{n\ell}(r) = \frac{u(r)}{r} = \frac{u(\rho)}{\rho/k} = \frac{k}{\rho}\left(\rho^{\ell+1} e^{-\rho} v(\rho)\right) = k\rho^\ell e^{-\rho} \sum_{j=-1}^{\infty} c_{j+1} \rho^{j+1}$$

$$= k(kr)^\ell e^{-kr} \sum_{j=0}^{\infty} c_j \rho^j$$

$k = \frac{\sqrt{-2mE}}{\hbar}$

(give me an ℓ, finding E and c_j's is solving problem).

$$c_{j+1} = \frac{2(j+\ell+1-n)}{(j+1)(j+2\ell+2)} c_j$$

$$E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2}$$ (gives k, $\phi_n(t)$, and helps determine when $\sum$ terminates).

Choose n, ℓ and start with an unknown c_0:

$$c_1 = c_{0+1} = \frac{2(0+\ell+1-n)}{(0+1)(0+2\ell+2)} c_0$$

$$c_2 = c_{1+1} = \frac{2(1+\ell+1-n)}{(1+1)(1+2\ell+2)} c_1 = [\ \ ][\ \ ]c_0$$

$\vdots$

$$c_{j_{max}+1} = \frac{2(j_{max}+\ell+1-n)}{\quad} c_{j_{max}}$$ ($c_{j_{max}} \neq 0$, $c_{j_{max}+1} = 0$)

$j_{max} + \ell + 1 - n = 0$

Degeneracy in ℓ comes from fixing n (E_n) and finding all pairs of j_{max} and ℓ which satisfy $j_{max}+\ell+1-n=0$

Each ℓ gives a different j_{max} and hence a different $v(\rho)$.

Question: Consider $\ell=1, n=3$ and $\ell=0, n=2$

Do these have the same ψ?

$v(\rho) = c_0 + c_1\rho$

$c_1^{31} = -\frac{2}{4}$ $\quad E_3 \Rightarrow k_3$

$c_1^{20} = -\frac{2}{2}$ $\quad E_2 \Rightarrow k_2$

also have ρ^ℓ in $R_{n\ell}$

furthermore Y_ℓ^m differ.

Just gimme the answer dammit!

$v(\rho) = L_{n-\ell-1}^{2\ell+1}(2\rho)$

$L_{q-p}^{p}(x) \equiv (-1)^p \left(\frac{d}{dx}\right)^p L_q(x)$ associated Laguerre polynomials

$L_q(x) \equiv e^x \left(\frac{d}{dx}\right)^q (e^{-x}x^q)$ q_{th} order Laguerre polynomial

Finally:

$$\psi_{n\ell m} = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}}\; e^{-r/na} \left(\frac{2r}{na}\right)^\ell L_{n-\ell-1}^{2\ell+1}\left(\frac{2r}{na}\right) Y_\ell^m(\theta,\phi)$$

$$E_n = -\frac{\hbar^2}{2ma^2}\frac{1}{n^2}$$

$a \equiv \frac{4\pi\epsilon_0\hbar^2}{me^2}$ = Bohr radius