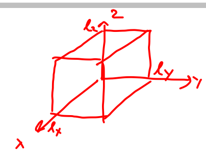
Free-electron Gas

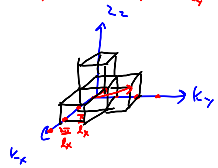
Today $\underbrace{N_q}_{\#\text{ of free electrons}} \equiv N_f$



$$\psi_{n_x n_y n_z} = \sqrt{\frac{8}{V}} \sin\left(\frac{n_x \pi}{l_x} x\right) \sin\left(\frac{n_y \pi}{l_y} y\right) \sin\left(\frac{n_z \pi}{l_z} z\right)$$

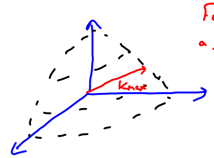
$$E_{n_x n_y n_z} = \frac{\hbar^2}{2m}\left(\frac{n_x^2 \pi^2}{l_x^2} + \frac{n_y^2 \pi^2}{l_y^2} + \frac{n_z^2 \pi^2}{l_z^2}\right) = \frac{\hbar^2}{2m} k^2$$

$$\vec{k} = \left(\frac{n_x \pi}{l_x}, \frac{n_y \pi}{l_y}, \frac{n_z \pi}{l_z}\right)$$



Volume in k-space associated with each additional state is $\frac{\pi^3}{V}$

Total amount of k-space occupied is $\frac{N_f}{2} \frac{\pi^3}{V}$



For large N_f this forms an $\frac{1}{8}$ of a sphere.

$$V_{oct} = \frac{1}{8}\left(\frac{4}{3}\pi k_{max}^3\right)$$

$$\frac{N_f}{2} \frac{\pi^3}{V} = \frac{1}{6}\pi k_{max}^3 \Rightarrow k_{max} \equiv k_F = (3\rho\pi^2)^{1/3}$$

ρ = free electron density $= \frac{N_f}{V}$

$$E_F = \frac{\hbar^2}{2m} k_F^2 = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3}$$

$$E_{tot} = \int dE = \int E(N)\, dN = \int \underbrace{E(N)}_{\frac{\hbar^2}{2m}k^2} \underbrace{\rho_k}_{\frac{V 2}{\pi^3}} \overbrace{dV_k}^{\frac{1}{8}(4\pi k^2)dk}$$

$$E_{tot} = \frac{\hbar^2}{2m} \frac{2V}{\pi^3} \frac{\pi}{2} \int_0^{k_F} k^4\, dk = \frac{\hbar^2 V}{10 m \pi^2} k_F^5 = \frac{\hbar^2 V}{10 m \pi^2}(3\rho\pi^2)^{5/3}$$

$\rho = \frac{N_f}{V}$

$$P = -\frac{dE_{tot}}{dV} = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3}$$

For bosons

$$E_{tot} = N_b \frac{\hbar^2 \pi^2}{2m}\left(\frac{1}{l_x^2} + \frac{1}{l_y^2} + \frac{1}{l_z^2}\right) = \frac{N_b 3\hbar^2\pi^2}{2m} V^{-2/3}$$

Assume $l_x = l_y = l_z$ $\quad V = l^3 \quad l^2 = V^{2/3}$

$$P_b = -\frac{dE_{tot}}{dV} = N_b \frac{\hbar^2\pi^2}{m} \frac{1}{V^{5/3}}$$

$$P_f = N_f^{5/3} \underbrace{\left(\frac{2}{5}\left(\frac{9}{\pi^2}\right)^{1/3}\right)}_{0.38} \frac{\hbar^2\pi^2}{2m} \frac{1}{V^{5/3}}$$